Robust and Accurate
Finite Difference Methods in Option Pricing
One Factor Models

Daniel J. Duffy

Contents

1 Introduction

This article describes a class of finite-difference schemes to approximate the solution of one-factor models in risk management applications. In particular, we apply so-called fitted difference schemes to European options. The proposed schemes do not suffer from the well-documented spurious oscillation problems that are seen in standard 'recipe-type' schemes. The schemes that we employ have been around since the early 1950's and were quite successful in fluid dynamics applications but were discarded because of the difficulty that human computers had in calculating the terms in the schemes. Other finite difference schemes (such as the famous Crank-Nicholson method) gained favour in that period and it would seem that most finite difference schemes in the financial risk literature use Crank-Nicholson or some variation thereof. The objective of this article is to show that the Crank-Nicholson method produces so-called spurious oscillations in the value of the derivative quantity and its sensitivities (such as the delta and gamma). We prove both mathematically and experimentally that these oscillations occur in Black-Scholes type equations with small volatility, large drift factor (or both). The oscillations are caused when centred differencing is used to approximate first-order derivatives in the underlying variable. The problem is resolved by the so-called monotone (exponentially fitted) difference schemes.

We structure this article as follows: in section 2 we give an overview of why we are using the finite difference method and what the alternative numerical techniques are. Section 3 lays the mathematical foundations for the derivatives whose behaviour can be described as parabolic initial boundary value problems (the so-called continuous problem). We take a partial differential equation approach and to this end we describe the behaviour of derivatives without having to resort to techniques such as Laplace and Fourier transforms, reduction to a simpler partial differential equation or Monte Carlo methods. Our feeling is that transforming the partial differential equation to some other form introduces new difficulties. We are interested in discovering and applying finite difference methods to the original partial differential equations and we wish to emulate the properties of the original continuous equations by the corresponding discrete ones. For the purposes of this article, we consider the maximum principle to be essential. This principle states that a problem with non-negative initial and boundary conditions has a solution with non-negative values. Finally, we discuss the different kinds of boundary conditions that we encounter in risk applications.

Section 4 is a critique of classical ('recipe-type') finite difference schemes and their application in risk. We discuss how these schemes deal with stability proofs and why they become unstable when the volatility is small or when the drift factor is large. We distinguish between semi-discrete schemes (in which the underlying variable is discretised and the time variable is kept continuous) and fully discrete schemes (where all variables are discrete) so that we can pinpoint when spurious oscillations occur and what the cause of these oscillations is. We shall see in particular that some popular classical schemes fail to satisfy a so-

called discrete variant of the maximum principle. Section 4 also introduces the so-called class of exponentially fitted schemes (see Doolan 1980) that will form the basic building blocks for the material in section 5.
Section 5 introduces a new class of robust and accurate finite difference methods for solving Black-Scholes equations. These methods are compared with classical centred-difference schemes (in particular, the Crank-Nicholson scheme). We first of all discretise in the space dimension in order to produce a so-called semi-discrete scheme. This scheme is equivalent to a system of first-order ordinary differential equations (ODES) with given initial conditions. We give necessary and sufficient conditions for the solution to be stable and non-oscillating. Once the semi-discrete scheme has been defined it can now be discretised in time by several time-stepping schemes such as Runge-Kutta, Crank-Nichsolson and the fully implicit scheme (see Crouzeix 1975).
Section 6 deals with the problem of approximating the derivatives of the solution of parabolic initial boundary value problems. In particular, we discuss how the various difference schemes approximate the option sensitivities (such as delta, gamma and rho). We shall see that large oscillation occur with the Crank-Nicholson method, especially when the option is at the money. There are no oscillations in the approximations to the greeks when the fitted method is used. An interesting way to approximate the option price and its delta directly is to convert the Black Scholes equation to a first order system and apply the so-called box scheme (see Keller 1970). This scheme avoids the oscillations that are found in the CN scheme, has second order accuracy and can be applied to problems with Robins boundary conditions.
Section 7 discusses the numerical implementation of the fitted schemes and how they compare with classical schemes such as Crank Nicholson.

2 Background and Assumptions

The Black Scholes equation was discovered in 1973 and since that time it has been used as a standard pricing formula for different kinds of options. The assumptions underlying this famous formula do not always hold and the original equation has been generalised to accommodate many new kinds of options. This means that an exact solution to the corresponding equation cannot always be found and we must then resort to approximate methods. The most popular techniques are:

Reduction of the equation to a simpler form (Wilmott 1993)
Monte Carlo and quasi Monte Carlo methods (Boyle 1977)
Binomial and trinomial methods (Cox 1985)
Reduction to other forms by means of Fourier transforms, for example (Carr 1998)

While each of the above approaches has its advantages and disadvantages we prefer to approximate the partial differential equations that model option prices by finite difference methods. Our reasons for taking this approach are:

The methods have a long history going back to the eighteenth century
They have a sound theoretical and mathematical basis
They are flexible and can be applied to many types of pricing problems
They can be easily programmed on a digital computer

Finite difference methods have been used by pure mathematicians to prove existence and uniqueness to solutions of boundary value problems. In the 1950's the methods proved to be of interest in their own right when engineers started to use them to solve engineering and scientific problems. The number of application areas is too great to enumerate but we mention a few: chemical reaction theory, fluid mechanics, semiconductor device modelling and magnetohydrodynamics. Our objective in this article is to show how finite difference methods can be applied to approximating the Black Scholes partial differential equation and its generalisations. In particular, we prove both mathematically and by numerical experiment that these methods are accurate, robust and are easy to design and to implement. Furthermore, we compare and contrast so-called classical difference schemes with a new family of difference schemes that perform well even when the parameters of the Black Scholes equation take on large and small values.

2.1 Critique of Finite Differences for Pricing Model

The application of finite difference theory is gaining in popularity in the financial world. The fact that many derivative types can be described unambiguously by partial differential equations or PDEs as they are sometimes called (such as the one-factor and multi-dimensional Black Scholes equations) has led a number of researchers and practitioners to investigate the possibilities of applying well-known finite difference schemes to such PDEs. The basic idea is easy to comprehend: just replace the continuous derivatives in the PDE by divided differences and solve the corresponding matrix system at each time level. The theory of PDE and finite differences goes as far back as the nineteenth century (and maybe even earlier!) and a number of branches of mathematics have grown around these. A number of such schemes have been applied to pricing problems and are the subject of some debate and research. Early sources are the textbooks by Hull, Wilmott and others (see Hull 1993, Wilmott 1993). The two main attention points with finite difference schemes are:

Proving that the difference scheme is stable (in some sense)
Proving that it converges to the true solution of the PDE

Most authors use the classic von Neumann amplification factor method to prove stability. This is a well-known technique but strictly speaking it can only be applied for PDEs with constant coefficients (thus not Black Scholes) and secondly it becomes difficult to prove stability for more complicated problems (see Zvan 1997). Furthermore, invoking the famous Lax equivalence theorem proves convergence to the exact solution of the PDE. This theorem states that under certain conditions stability of a difference scheme is the only requirement for convergence (see Richtmeyer and Morton 1967). Unfortunately, 'real-life' PDEs are notoriously difficult to solve numerically and the Lax theorem is not of much help. The problems that are we are faced with are numerous but we give a list of the main issues that must be resolved if we are to be successful in producing robust and reliable schemes:

- We need new techniques for proving stability of schemes that approximate PDEs having non-constant, nonlinear and discontinuous coefficients. Furthermore, these techniques must be able to handle Dirichlet, Neumann and mixed boundary conditions.
- Higher-order PDEs may degenerate into lower-order PDEs under certain circumstances (for example, with exponentially decaying volatilities or large drift factors in the Black Scholes equations). We describe these equations as being of singular perturbation type and these have been studied in great detail, both theoretically and numerically (see Vishik 1957, Doolan 1980, Duffy 1980).
- A number of finite difference schemes are too elaborate or complicated. For example, numerical schemes that are nonlinear in nature while the corresponding PDEs are linear deserve special attention because they may be 'overkill' in our opinion. Finally, many difference schemes use advanced direct and iterative matrix techniques to solve the discrete system of equations. This is again overkill because tridiagonal matrix systems can be solved by more efficient methods.

The objective is to show how to define robust and reliable difference schemes for the Black-Scholes equation and its generalistions while avoiding some of the pitfalls of classical finite difference schemes. Much of this work is based on the author's research in the area of numerical analysis.

3 The Continuous Problem

We introduce the basic set of equations that model the behaviour of a class of derivative products. In particular, we model derivatives that are described by so-called initial boundary value problems of parabolic type (see Il'in 1962). To this end, consider the general parabolic equation

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u - \frac{\partial u}{\partial t} = f(x,t) \qquad (1)$$

where the functions a_{ij}, b_j, c and f are real and take finite values, $a_{ij} = a_{ji}$, and

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\alpha_i\alpha_j > 0 \qquad \text{if} \qquad \sum_{j=1}^{n} \alpha_j^2 > 0 \qquad (2)$$

In (2) the variable x is a point in n-dimensional space and t is considered to be a positive time variable. Equation (1) is the general equation that describes the behaviour of many derivative types. For example, in the one-dimensional case ($n = 1$) it reduces to the famous Black-Scholes equation (see Black 1973)

$$\frac{\partial V}{\partial t^\star} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0 \tag{3}$$

where V is the derivative type, S is the underlying asset (or stock), σ is the constant volatility, r is the interest rate and D is a dividend. Equation (3) can be generalised to the multivariate case

$$\frac{\partial V}{\partial t^\star} + \sum_{j=1}^{n}(r - D_j)S_j\frac{\partial V}{\partial S_j} + \frac{1}{2}\sum_{i,j=1}^{n}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} = rV \tag{4}$$

(see Bhansali 1998). This equation models a multi-asset environment. In this case σ_i is the volatility of the i^{th} asset and ρ_{ij} is the correlation between assets i and j. In this case we see that the local change in time (namely the factor $\frac{\partial V}{\partial t^\star}$) is written as the sum of three terms:

Interest earned on cash position

$$r\Big(V - \sum_{j=1}^{n} S_j \frac{\partial V}{\partial S_j}\Big) \tag{5}$$

Gain from dividend yield

$$\sum_{j=1}^{n} D_j S_j \frac{\partial V}{\partial S_j} \tag{6}$$

Hedging costs or slippage

$$-\frac{1}{2}\sum_{i,j=1}^{n}\rho_{ij}\sigma_i\sigma_j \frac{\partial^2 V}{\partial S_i \partial S_j} \tag{7}$$

Our interest in this article is in discovering and documenting robust numerical schemes that produce reliable and accurate results irrespective of the size of the parameter values in (4). We restrict ourselves to the single-factor case ($n = 1$). Multi-factor models and their numerical approximations by finite difference methods will be discussed elsewhere. Equation (1) has an infinite number of solutions in general. In order to reduce this number to one, we need to define some constraints. To this end, we define so-called initial condition and boundary conditions for (1). We achieve this by defining the space in which equation (4) is assumed to be valid. Since the equation has a second order derivative in x and a first order derivative in t we should expect that a unique solution can be found by defining two boundary conditions and one initial condition (we must mention that risk applications tend to work with so-called terminal conditions while researchers in science and engineering tend to work with initial conditions). In general, we note that there are three types of boundary conditions associated with equation (1) (see Il'in 1962). These are:

First boundary value problem (Dirichlet problem)
Second boundary value problem (Neumann,Robins problems)
Cauchy problem

The first boundary value problem is concerned with the solution of (1) in a domain $D = \Omega X(0,T)$ where Ω is a bounded subset of $\mathbb{R}^{\mathrm{n}}$ and T is a positive number. In this case we seek to find a solution of (1) satisfying the conditions

$$\begin{aligned} u|_{t=0} &= \varphi(x) \qquad \text{(initial condition)} \\ u|_{\Gamma} &= \psi(x,t) \qquad \text{(boundary condition)} \end{aligned} \tag{8}$$

where Γ is the boundary of Ω . The boundary conditions in (8) are sometimes called Dirichlet boundary conditions. These conditions arise when we model single and double barrier options in the one-factor case (see Topper 2000). They also occur when we model European options. The second boundary value problem is similar to (8) except that instead of giving the value of u on the boundary Γ the directional derivatives are included, as seen in the following specification:

$$\left(\frac{\partial u}{\partial \eta} + a(x,t)u\right)|_{\Gamma} = \psi(x,t) \tag{9}$$

In this case $a(x,t)$ and $\psi(x,t)$ are known functions of x and t, and $\frac{\partial}{\partial \eta}$ denotes the derivative of u with respect to the outward normal η at Γ . A special case of (9) is when $a(x,t) \equiv 0$; then (9) represents the so-called Neumann boundary conditions. These occur when modelling certain kinds of put options. Of course (9) should be augmented by an initial condition similar to that in equation (8). Finally, the solution of the Cauchy problem for (1) in the strip $\mathbb{R}^n X(0,T)$ is given by the initial condition

$$u|_{t=0} = \varphi(x) \tag{10}$$

where $\varphi(x)$ is a given continuous function and $u(x,t)$ is a function that satisfies (1) in $\mathbb{R}^n X(0,T)$ and satisfies the initial condition (10). This problem allows negative values of the independent variable $x = (x_1, \ldots, x_n)$. A special case of the Cauchy problem can be seen in the modelling of one-factor European and American options (see Wilmott 1993) where x is represented by the underlying asset S. Boundary conditions are given by values at $S = 0$ and $S = \infty$. For European options these conditions are:

$$\begin{aligned} &C(0,t) = 0 \\ &C(S,t) \to S \qquad \text{as} \qquad S \to \infty \end{aligned} \tag{11}$$

Here C (the role played by u in equation (1)) is the variable representing the price of the call option. For European put options, on the other hand the boundary conditions are:

$$\begin{aligned} &P(0,t) = Ke^{-r(T-t^{\star})} \\ &P(S,t) \to 0 \qquad \text{as} \qquad S \to \infty \end{aligned} \tag{12}$$

Here P (the role played by u in equation (1) is the variable representing the price of the put option, K is the strike price, r is the risk-free interest rate, T is the time to expiry and t is the current time. In practice, we see that many articles solve European options problems numerically by assuming a finite domain, that is one in which the right-hand boundary conditions in (11) and (12) are defined at large but finite values of S.The rest of this article concentrates on one-factor models and we adopt the notation that is used in engineering and scientific applications. For example, the variable $t = T - t^\star$ is used as the independent time variable while risk researchers tend to work with the variable $t^\star$ in their equations. Thus, financial engineers should be aware of the fact that we are solving initial boundary value problems rather than terminal boundary value problems. Thus, from this point on we assume the following 'canonical' form for the operator L in equation (1):

$$Lu \equiv -\frac{\partial u}{\partial t} + \sigma(x,t)\frac{\partial^2 u}{\partial x^2} + \mu(x,t)\frac{\partial u}{\partial x} + b(x,t)u = f(x,t) \tag{13}$$

where σ, μ and b are known functions of x and t. We place no continuity or boundedness conditions on them for the moment.

3.1 The Maximum Principle for Parabolic Equations

We now introduce a number of key results that describe how possible solutions of (1) (and its one-factor equivalent subcase (13)) behave in the domain of interest. In particular, we discuss the continuous maximum principle that states the conditions under which the solution of (1) remains positive when its corresponding initial and boundary conditions are positive. Let $D = \Omega X(0,T)$ and let $\bar{D}$ be its closure. The following results are proved in Il'in 1962 and their discrete equivalents will be introduced in sections 4 and 5.

Theorem 1 *Assume that the function $u(x,t)$ is continuous in D and assume that the coefficients in (13) are continuous. Suppose that $Lu \leq 0$ in $\bar{D}\backslash\Gamma$ where $b(x,t) < M$ (M is some constant) and suppose furthermore that $u(x,t) \geq 0$ on Γ. Then*

$$u(x,t) \geq 0 \qquad in \qquad \bar{D}.$$

This theorem states that positive initial and boundary conditions lead to a positive solution in the interior of the domain D. This has far-reaching consequences as we shall see later in this article.

Theorem 2 *Suppose that $u(x,t)$ is continuous and satisfies (13) in $\bar{D}\backslash\Gamma$ where $f(x,t)$ is a bounded function ($|f| \leq N$) and $b(x,t) \leq 0$. If $|u(x,t)|_\Gamma \leq m$ then*

$$|u(x,t)| \leq Nt + m \qquad in \qquad \bar{D} \tag{14}$$

We can sharpen the results of Theorem 2 in the case where $b(x,t) \leq b_0 < 0$. In this case estimate (14) is replaced by

$$|u(x,t)| \leq max\left\{\frac{-N}{b_0}, m\right\} \tag{15}$$

*P*roof: Define the so-called 'barrier' function $w^{\pm}(x,t) = N_1 \pm u(x,t)$ where $N_1 = max\left\{\frac{-N}{b_0}, m\right\}$. Then $w^{\pm} \geq 0$ and $Lw^{\pm} \leq 0$.
By Theorem 1 we deduce that $w^{\pm} \geq 0$ in $\bar{D}$. The desired result follows.

The inequality (15) states that the growth of u is bounded by its initial and boundary values. It is interesting to note in the special case $b \equiv 0$ and $f \equiv 0$ that we can deduce the following maximum and minimum principles for the heat equation and its variants.

Corollary 1 *Assume that the conditions of Theorem 2 are satisfied and that $b \equiv 0$ and $f \equiv 0$. Then the solution $u(x,t)$ takes its least and greatest values on Γ, that is*

$$m_1 = min\ u(x,t) \leq u(x,t) \leq max\ u(x,t) \equiv m_2$$

The results from Theorems 1 and 2 and Corollary 1 are very appealing: you cannot get negative values of the solution u from positive input. We shall produce similar results for the finite difference analogues of the operator L in (13) and we shall show how it is possible to prove stability by a discrete maximum principle without having to resort to the more restricted von Neumann stability technique. We now conclude this section with a result that is of particular importance when we are interested in proving positivity of the solutions of (13) in unbounded domains, for example, European and American option problems.

Theorem 3 *(Maximum principle for the Cauchy problem)*
Let $u(x,t)$ be continuous and bounded below in $H = \mathbb{R}^n X(0,T)$, that is $u(x,t) \geq -m, m > 0$. Suppose further that $u(x,t)$ has continuous derivatives in H up to second order in x and first order in t and that $Lu \leq 0$. Let σ, μ and b satisfy

$$|\sigma(x,t)| \leq M(x^2+1)$$
$$|\mu(x,t)| \leq M\sqrt{x^2+1}$$
$$b(x,t) \leq M$$

Then $u(x,t) \geq 0$ everywhere in H if $u \geq 0$ for $t = 0$.

We can apply Theorem 3 to the one-factor Black-Scholes equation (3) in order to convince ourselves that the price of an option can never take negative values.

3.2 Some Special Cases

This article focuses on a specific problem, namely the one-factor generalized Black-Scholes equation with initial condition and Dirichlet boundary conditions. Define $\Omega = (A, B)$ where A and B are two real numbers.
The formal statement of the problem is:

Find a function $u : D \to \mathbb{R}^1 (D = \Omega X(0,T))$ such that

$$Lu \equiv -\frac{\partial u}{\partial t} + \sigma(x,t)\frac{\partial^2 u}{\partial x^2} + \mu(x,t)\frac{\partial u}{\partial x} + b(x,t)u = f(x,t) \;\; in \;\; D \quad (16)$$

$$u(x,0) = \varphi(x),\; x \in \Omega \quad (17)$$

$$u(A,t) = g_0(t),\; u(B,t) = g_1(t),\; t \in (0,T) \quad (18)$$

The initial-boundary value problem (16)-(18) is very general and it subsumes may specific cases from the risk literature (in particular it is a generalization of the original Black-Scholes equation).

In general, the coefficients $\sigma(x,t)$ and $\mu(x,t)$ represent volatility (diffusivity) and drift (convection), respectively. Equation (16) is called the convection-diffusion and it has been the subject of much study. It serves as a model for many kinds of physical and economic phenomena. Much research has been carried out in this area, both on the continuous problem and its discrete formulation (for example, using finite difference and finite element methods). In particular, research has shown that standard centred-difference schemes fail to approximate (16)-(18) properly in certain cases (see Farrell 2000). The problems are well-known in the scientific and engineering worlds. Some risk modelers apply finite difference schemes in a 'cookbook' fashion without being aware of the fact that standard schemes do not always work well for convection-diffusion equations. Another source of problems is that many textbooks do not tackle (16)-(18) head-on but instead content themselves with devising finite difference schemes for the heat equation. Classical schemes perform well in this latter case but they cannot be extrapolated to the convection-diffusion equation. We aim to show that classical finite difference method do not always produce good results for (16)-(18)) and why these schemes are not good. Furthermore, we devise a family of robust and accurate finite difference schemes for this problem.

We now investigate some special limiting cases in the system (16)-(18). One particular case is when the function $\sigma(x,t)$ tends to zero as a function of x or t (or both). The classic Black Scholes equation assumes that volatility is constant but this is not always true in practice. For example, the volatility may be time-dependent (see Wilmott 1993). In general, the volatility may be a function of both time and the underlying variable. If the volatility is a function of time only then an explicit solution can be found but an explicit solution cannot be found in more complicated cases. We note the so-called exponentially declining volatility functions (see van Deventer 1997) as given by the formula

$$\sigma(t) = \sigma_0 e^{-\alpha(T-t)} \quad (19)$$

where σ_0 and α are given constants.

Having described situations in which the coefficient of the second derivative in equation (16) can be small or tend to zero we now discuss what the mathematical implications are. This is very important in general because finite difference schemes must be robust enough to model the exact solution in all extreme cases. Setting σ to zero in (16) leads us to a formally first-order hyperbolic equation

$$L_1 u \equiv -\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + b(x,t)u = f(x,t) \quad (20)$$

Since the second derivative in x is not present in (20) we conclude that only one boundary condition and one initial condition is needed in order to specify a unique solution (see Friedrichs 1958, Duffy 1977). But the question is which boundary condition in (18) should we choose? In order to answer this question we must define the so-called characteristic lines associated with equation (20) (see Godounov 1979, Godounov 1973). These are defined as lines that satisfy the ordinary differential equation

$$\frac{dx}{dt} = -\mu \tag{21}$$

The lines have positive or negative slope depending on whether μ has negative or positive values. In general, it can be shown (see Friedrichs 1958) how to discover the 'correct' boundary condition for (20):

$$\begin{aligned} u(A,t) &= g_0(t) \qquad \text{if} \qquad \mu < 0 \\ u(B,t) &= g_1(t) \qquad \text{if} \qquad \mu > 0 \end{aligned} \tag{22}$$

We see that one of the boundary conditions in (18) is superfluous and this situations leads to the well-known boundary layer phenomenon that is seen in fluid dynamics problems (See Vishik 1957 for a fundamental analysis of ordinary and partial differential equations in which the coefficient of the highest-order derivative is small or tends to zero as $t \to \infty$).
We thus see that as $\sigma \to 0$ that we cannot hope to satisfy both boundary conditions in (18). The presence of the resulting boundary layer causes major problems for classical finite difference methods because these are unable to cope with the approximation of the exact solution in the boundary layer itself. We shall quantify this in more detail in section 4. In general, we say that system (16)-(18) is of singular perturbation type when either σ is small or μ is large. We sometimes say that the system is convection-dominated in the latter case.
Another special case in (16) is when $\mu \to 0$ (or is zero). The resulting equation is similar to the heat equation and it can be reduced to the heat equation by a clever change of variables (see Wilmott 1993). This is a well-documented equation and classical difference schemes approximate it well. The challenge in this article however, is to devise finite difference schemes that work under all conditions irrespective of the values of the parameters in equation (16).
We finish this section with a remark that the above discussion can be applied to two-factor models of the form:

$$-\frac{\partial u}{\partial t} + \sigma_1 \frac{\partial^2 u}{\partial x^2} + \sigma_2 \frac{\partial^2 u}{\partial y^2} + \mu_1 \frac{\partial u}{\partial x} + \mu_2 \frac{\partial u}{\partial y} + bu = f \tag{23}$$

This equation occurs when modeling two-factor Gaussian term structures (see Levin 2000). In this case σ_1 is the short-rate volatility constant, σ_2 is the long yield volatility and μ_1, μ_2 are given in terms of other known parameters. The reduced equation is given by

$$-\frac{\partial u}{\partial t} + \mu_1 \frac{\partial u}{\partial x} + \mu_2 \frac{\partial u}{\partial y} + bu = f \tag{24}$$

with the resulting loss of boundary conditions. Equation (24) is called hyperbolic (in the sense of Friedrichs 1958). Classical difference schemes will be no better at approximating equations (23) and (24) than in the one-dimensional case; again, robust methods are needed. A discussion of such methods is beyond the scope of this article.

4 Finite Difference Methods for Risk Management: Motivation

This section deals with numerical methods for what could be called the 'time-independent' Black-Scholes equation. This is the Black-Scholes equation with the dependencies on time removed. This is in fact a so-called two-point boundary value problem. There are three reasons for taking this approach. First, it is easier to solve than the full-blown Black-Scholes equation and we can motivate how and why classical difference schemes do not always work when the coefficient of the first derivative is large. In particular, we can predict when bounded and unbounded oscillations in the approximate solution occur. Second, we introduce and motivate a class of so-called 'fitted' difference schemes that are unconditionally stable and converge to the exact solution of the corresponding two-point boundary value problem. Furthermore, the fitted schemes never oscillate. Finally, the fitted schemes will be adapted to solve the full Black-Scholes equation by first discretising in the direction of the underlying S (the so-called semi-discrete scheme) and then in the time direction (using time-stepping methods such as Crank-Nicholson and Runge-Kutta). The corresponding scheme is unconditionally stable and convergent and no oscillations are to be found (the proof of these results is to be found in Duffy 1980). This means that the method works regardless of the size of the volatility, value of the underlying or interest rate.

It is interesting to note that fitted methods have been in existence since the 1950's and much work has been done in the area. In particular, a group of Irish and Russian mathematicians have done fundamental work on robust finite difference schemes for boundary value problems containing large and small coefficients. For more information, see de Allen 1955, Doolan 1980 and Farrell 2000.

4.1 Notation and some Classical Finite Difference Schemes

Our aim is to approximate (16)-(18) by finite difference schemes. To this end, we divide the interval $[A, B]$ into the sub-intervals

$$A = x_0 < x_1 < \ldots < x_J = B$$

and we assume for convenience that the mesh-points $\{x_j\}_{j=0}^{J}$ are equidistant, that is

$$x_j = x_{j-1} + h, \quad j = 1, \ldots, \quad J \ (h = \tfrac{B-A}{J})$$

Furthermore, we divide the interval $[0, T]$ into N equal sub-intervals

$$0 = t_0 < t_1 < \ldots < t_N = T$$

where

$$t_1 = t_{n-1} + k, \quad n = 1, \ldots, N \quad (k = \tfrac{T}{N})$$

(It is possible to define non-equidistant mesh-points in the x and t directions but doing so would complicate the mathematics and we would be in danger of losing focus.)
The essence of the finite difference approach lies in replacing the derivatives in (16) by divided differences at the mesh-points (x_j, t_n) We define the difference operators in the x-direction as follows:

$$\begin{aligned} &D_+ u_j = (u_{j+1} - u_j)/h, \quad D_- u_j = (u_j - u_{j-1})/h \\ &D_0 u_j = (u_{j+1} - u_{j-1})/2h, \quad D_+ D_- u_j = (u_{j+1} - 2u_j + u_{j-1})/h^2 \end{aligned}$$

It can be should by Taylor expansions that D_+ and D_- are first-order approximations to $\frac{\partial}{\partial x}$ while D_0 is a second-order approximation to $\frac{\partial}{\partial x}$. Finally, $D_+ D_-$ is a second-order approximation to $\frac{\partial^2}{\partial x^2}$
We define approximations to $\frac{\partial u}{\partial t}$ in section 5.
We now consider the following two-point boundary-value problem (TPBVP): find a function u such that

$$\begin{aligned} &\sigma \frac{d^2 u}{dx^2} + 2\frac{du}{dx} = 0 \qquad \text{in} \qquad (0,1) \\ &u(0) = 1, \quad u(1) = 0 \end{aligned} \tag{25}$$

We assume that σ is a positive constant for the moment. We now replace the derivatives in (25) by their corresponding finite differences in order to produce the following finite difference method: find a mesh-function $\{U_j\}_{j=1}^{J-1}$ such that

$$\begin{aligned} &\sigma D_+ D_- U_j + 2D_0 U_j = 0, \quad j = 1, \ldots, J-1 \\ &U_0 = 1, \quad U_J = 0 \end{aligned} \tag{26}$$

Scheme (26) is sometimes called the divided difference scheme and it is a standard way to approximate convection-diffusion equations. In particular, many risk authors use it to approximate the Black-Scholes equation. We now show that (26) does not always produce good results!
It can be checked that the exact solution of (25) is given by

$$u(x) = \frac{e^{-2x/\sigma} - e^{-2/\sigma}}{1 - e^{-2/\sigma}} \tag{27}$$

It can also be checked that the exact solution of (26) is given by

$$U_j = (\lambda^j - \lambda^J)/(1 - \lambda^J), \qquad where \qquad \lambda = \frac{1 - h/\sigma}{1 + h/\sigma} \tag{28}$$

(see Farrell 2000 for more details).
Let us now assume that $\sigma < h$ in equation (28). This means that $\lambda < 0$ and thus λ^j is positive or negative depending on whether j is even or odd! Furthermore,

$$\lim_{\sigma \to 0} U_j = ((-1)^j + 1)/2$$

Hence U_j oscillates in a bounded fashion for all σ satisfying $\sigma < h$ We thus conclude that centred finite difference schemes are unsuitable for the numerical solution of problem (25) when $\sigma < h$. It can even be shown that this difference scheme produces a solution that goes to infinity as $\sigma \to 0$ tends to zero (see Farrell 2000, page 18). This very simple one-dimensional example leads us to conclude that similar problems will (and are) experienced with standard recipe-type methods (for example, as used in Press 1989). Thus, we rule out these schemes as a viable option for solving generalised Black-Scholes equations.
Is there hope? As an alternative to centred divided differences we could approximate (25) by so-called upward finite difference schemes:

$$\begin{aligned} &\sigma D_+ D_- U_j + 2D_+ U_j, \quad j = 1, \ldots, J-1 \\ &U_0 = 1, \;\; U_J = 0 \end{aligned} \tag{29}$$

The solution of (29) is given by

$$U_j = \frac{\lambda^j - \lambda^J}{1 - \lambda^J}, \quad \lambda = \frac{1}{1 + \frac{2h}{\sigma}}$$

We note that

$$U_1 - u(x_1) = \frac{\lambda - \lambda^J}{1 - \lambda^J} - \frac{r - r^J}{1 - r^J}, \quad r = e^{-2h/\sigma}$$

where $u = u(x)$ is the solution of defined by (29). We now set $\frac{\sigma}{h} = 1$ and we let $N \to \infty$. Then

$$U_1 - u(x_1) = \frac{1}{3} - e^{-2} = 0.197998$$

This means that the maximum pointwise error is 20% no matter what the size of h is. This is clearly unacceptable! Our conclusion is as follows: centred divided or one-sided difference schemes will result in either spurious oscillations and/or an inaccurate solution. The situation is no better with classical finite element schemes and we rule them out as well (for a discussion of fitted finite element schemes see Ikeda 1983 and Topper 2000 for results on the application of finite element packages to the solution of exotic options).

4.2 A new Class of Robust Difference Schemes

We now introduce the class of exponentially fitted schemes for general two-point boundary value problems and we apply these schemes to the Black-Scholes equation. We lay the foundations for the rest of this article here.

Exponentially fitted schemes are stable, have good convergence properties and do not produce spurious oscillations. In order to motivate what an exponentially fitted difference scheme is, let us look at the slightly more general TPBVP:

$$\begin{aligned}&\sigma\frac{d^2u}{dx^2}+\mu\frac{du}{dx}=0 \qquad in \qquad (A,B)\\ &u(A)=\beta_0,\quad u(B)=\beta_1\end{aligned} \tag{30}$$

Here we assume that σ and μ are positive constants. We now approximate (30) by the difference scheme defined as follows:

$$\begin{aligned}&\sigma\rho D_+D_-U_j+\mu D_0U_j=0,\quad j=1,\dots,J-1\\ &U_0=\beta_0,\quad U_J=\beta_1\end{aligned} \tag{31}$$

where ρ is a so-called fitting factor (this factor is identically equal to 1 in the case of the centred difference scheme (26)). We now choose ρ so that the solutions of (30) and (31) are identical at the mesh-points. Some easy arithmetic shows that

$$\rho=\frac{\mu h}{2\sigma}\coth\frac{\mu h}{2\sigma}$$

where $\coth x$ is the hyperbolic cotangent function defined by

$$\coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}}=\frac{e^{2x}+1}{e^{2x}-1}$$

The fitting factor will be used when developing fitted difference schemes for even more general TPBVPs. In particular, we discuss the following problem:

$$\begin{aligned}&\sigma(x)\frac{d^2u}{dx^2}+\mu(x)\frac{du}{dx}+b(x)u=f(x)\\ &u(A)=\beta_0,\quad u(B)=\beta_1\end{aligned} \tag{32}$$

where σ,μ and b are given continuous functions, and

$$0\le\sigma(x),\quad \mu(x)\ge\alpha>0,\quad b(x)\le 0\ \text{ for } x\in(A,B)$$

The fitted difference scheme that approximates (32) is defined by:

$$\begin{aligned}&\gamma_j^hD_+D_-U_j+\mu_jD_0U_j+b_jU_j=f_j,\quad j=1,\dots,J-1\\ &U_0=\beta_0,\quad U_J=\beta_1\end{aligned} \tag{33}$$

where

$$\begin{aligned}&\gamma_j^h=\frac{\mu_jh}{2}\coth\frac{\mu_jh}{2\sigma_j}\\ &\sigma_j=\sigma(x_j),\quad \mu_j=\mu(x_j),\quad b_j=b(x_j)\end{aligned} \tag{34}$$

We now state the following fundamental results (see Il'in 1969, Doolan 1980, Duffy 1980).

Theorem 4 *(Uniform Stability)*
The solution of scheme (33) is uniformly stable, that is

$$|U_j| \leq |\beta_0| + |\beta_1| + \frac{1}{\alpha} max_{k=1,\ldots,J} |f_k|, \quad j = 1, \ldots, J-1$$

Furthermore, scheme (33) is monotone in the sense that the matrix representation of (33)

$$AU = F$$

where $U = {}^t(U_1, \ldots, U_{J-1}), \quad F = {}^t(f_1, \ldots, f_{J-1})$ *and*

$$\mathbf{A} = \begin{pmatrix} \ddots & & \ddots & & 0 \\ & \ddots & & a_{j,j+1} & \\ \ddots & & a_{j,j} & & \ddots \\ & a_{j,j-1} & & \ddots & \\ 0 & & \ddots & & \ddots \end{pmatrix}$$

$$\begin{aligned} a_{j,j-1} &= \frac{\gamma_j^h}{h^2} - \frac{\mu_j}{2h} > 0 \qquad always \\ a_{j,j} &= -\frac{2\gamma_j}{h^2} + b_j < 0 \qquad always \\ a_{j,j+1} &= \frac{\gamma_j}{h^2} + \frac{\mu_j}{2h} > 0 \qquad always \end{aligned} \tag{35}$$

produces positive solutions from positive input.

Sufficient conditions for a difference scheme to be monotone have been investigated by many authors in the last 30 years; we mention the work of Samarski 1976 and Stoyan 1979. The latter author has used several various fitting factors:

$$\begin{aligned} \rho_0 &= \sigma^{-1}\{1 + q^2/(1 + |q|)\} \\ \rho_1 &= (1 + q^2)^{1/2} \\ \rho_2 &= \sigma^{-1}(\gamma) \end{aligned} \tag{36}$$

where γ is the fitting factor in (34).

Stoyan also produced stable and convergent difference schemes for the convection-diffusion equation producing results and conclusions that are similar to the present author's work (see Duffy 1980).

Theorem 5 *(Uniform Convergence)*
Let u *and* U *be the solutions of (32) and (33), respectively. Then*

$$|u(x_j) - U_j| \leq Mh$$

where M *is a positive constant that is independent of* h *and* σ.

The conclusion is that fitted scheme (33) is stable, convergent and produces no oscillations for all parameter regimes. In particular, the schemes 'degrades gracefully' to a well-known stable schemes when σ tends to zero.

5 Robust Finite Difference Methods

This section builds on section 4 by constructing fitted difference schemes for the convection-diffusion equation in general and the Black-Scholes equation in particular. Our tactic is to discretise in the space (or underlying) direction first, thus producing a system of ordinary differential equations (ODE). Having done that, we use two-level time-stepping (for example, Crank Nicholson) to discretise the ODE. The resulting linear system of equations can then be solved using standard matrix analysis at each time level.
We discuss a specific difference scheme that was proposed in Duffy 1980 and we state the main results concerning its stability and convergence properties. Finally, we discuss how the fitted scheme degrades gracefully to a well-known scheme when the coefficient of the second derivative tends to zero.

5.1 Semi-discrete Schemes

We define a semi-discrete scheme as one where the x (or S) direction is discretised while the time direction is continuous. We thus replace the x-derivatives in (16) by their corresponding divided differences. The corresponding semi-discrete scheme is:

$$\begin{aligned}
&-\frac{dU_j}{dt} + \gamma_j D_+ D_- U_j + \mu_j D_0 U_j + b_j U_j = f_j, \quad j = 1, \dots, J-1, \quad t \in (0,T) \\
&U_0 = g_0, \quad U_J = g_1 \\
&U_j(0) = \varphi_j, \quad j = 1, \dots, J-1
\end{aligned} \tag{37}$$

This scheme can be written as a linear system

$$\begin{aligned}
-\frac{dU}{dt} + AU &= F \\
U(0) &= U_0
\end{aligned} \tag{38}$$

Here U is the unknown vector ${}^t(U_1, \dots, U_{J-1}), U_0 = {}^t(\varphi_1, \dots, \varphi_{J-1})$ and F is the vector of right-hand side coefficients (note that this incorporates the boundary conditions) and the matrix A is given by equation (35).

We assume that A and F are independent of time in section 5.1 anf 5.2. In section 5.3 we deal with time-dependent coefficients.

We are interested in studying the stability of the system (38) as a function of the right-hand terms and initial conditions. To this end, we must examine the properties of the matrix A as defined by (35). We say that A is irreducible if its directed graph is strongly connected. An equivalent statement is that A has non-vanishing off-diagonal elements. We say that A is an M-matrix (with $a_{ij} \leq 0 \; \forall i \neq j$) if A is non-singular and a sufficient condition for A to satisfy $A^{-1} > 0$ is that $a_{ij} \leq 0 \, \forall i \neq j$ and $a_{ii} > 0, \quad i = 1, \dots, J-1$ (for a proof see Varga 1962).

Theorem 6 *(Limit Theorem)*
Let A be an irreducible M matrix having n rows and n columns. Then the unique solution of (38) is uniformly bounded in norm for all $t \geq 0$ and satisfies

$$\lim_{t \to \infty} U(t) = A^{-1} F$$

Corollary 2 *The semi-discrete scheme (38) based on the de Allen/Southwell/Il'in fitting operator satisfies the conditions of Theorem 6.*

Proof: See Varga 1962.

We are interested in determining the conditions under which spurious oscillations occur in the semi-discrete scheme (38). Most of the problems are caused by the eigenvalues of A.

Definition: A real matrix $Q = (q_{ij})$ is said to be essentially positive if $q_{ij} \geq 0$ and Q is irreducible.

Theorem 7 *Let Q be an essentially positive matrix. Then Q has a real eigenvalue $\lambda(Q)$ such that*

1) there exists an eigenvector $x > 0$ corresponding to $\lambda(Q)$
2) If α is another eigenvalue of Q then $Re\, \alpha < \lambda(Q)$
3) $\lambda(Q)$ increases when an element of Q increases

Theorem 8 *(Asymptotic behaviour)*
Let Q be an $n \times n$ essentially positive matrix. If $\lambda(Q)$ is the eigenvalue of Theorem 7 then

$$||exp\,(tQ)|| \leq K\, exp\,(t\lambda(Q)), \quad t \to \infty$$

where K is a positive constant independent of t.

Thus $\lambda(Q)$ dictates asymptotic behaviour of $||exp\,(tQ)||$ for large t.

Definition: Let Q be essentially positive. Then Q is called:

Supercritical if $\lambda(Q) > 0$
Critical if $\lambda(Q) = 0$
Subcritical if $\lambda(Q) < 0$

We now consider the problem

$$\begin{aligned} \frac{dU}{dt} &= QU(t) + r \ in \ (0, T) \\ U(0) &= U_0 \end{aligned} \tag{39}$$

Theorem 9 *(Asymptotic behaviour of solution)*
Let Q be essentially positive and non-singular. If Q is supercritical then for a given initial vector U_0 the solution of (39) satisfies

$$\lim_{t\to\infty} ||U(t)|| = \infty.$$

If Q is subcritical then $U(t)$ is uniformly bounded in norm for all $t > 0$ and satisfies

$$\lim_{t\to\infty} U(t) = -Q^{-1}r.$$

We thus see that it is necessary to have negative eigenvalues if we wish to ensure stable asymptotic behaviour of the solution of (39).

We give an example in the scalar case to motivate Theorem 9. Consider the simple initial value problem

$$\begin{aligned} \frac{du}{dt} &= qu + r, \quad t > 0 \\ u(0) &= A \end{aligned}$$

where q and r are constant. By using the integrating factor method, we can show that the solution is given by

$$u(t) = Ae^{qt} - \frac{r}{q}[1 - e^{qt}]$$

Thus, if $q < 0$ (the subcritical case) we see that

$$\lim_{t\to\infty} u(t) = -\frac{r}{q}$$

while if $q > 0$ (the supercritical case) it is unbounded.

Finally, if $q \equiv 0$ the solution is given by

$$u(t) = A + rt \ \text{(linear growth)}.$$

We now show how centred difference schemes produce supercritical matrices (and hence can grow unboundedly in time). To this end, we take $\sigma = 1$, μ constant and $f \equiv b \equiv 0$ for convenience. The semi-discrete scheme in this case is

$$-\frac{dU_j}{dt} + D_+D_-U_j + \mu D_0 U_j = 0 \tag{40}$$

or written in system form

$$\frac{dU}{dt} = AU \tag{41}$$

where $A = (a_{ij})$ with

$$\begin{aligned} a_{i,i-1} &= h^{-2} - \frac{\mu}{2h} \\ a_{ii} &= -2/h^2 \\ a_{i,i+1} &= h^{-2} + \frac{\mu}{2h} \end{aligned}$$

It can be shown (see Lam 1976) that the eigenvalues of A are given by

$$\lambda_m = a_{i,i} + 2(a_{i,i-1}a_{i,i+1})^{1/2}\cos\varphi_m, \quad \varphi_m = m\pi/J$$

and the corresponding eigenvectors are given by

$$y_j^m = \left(\frac{a_{i,i-1}}{a_{i,i+1}}\right)^{1/2}\sin(j\varphi_m)$$

We thus see that the eigenvalues of A are in fact complex if $a_{i,i-1} < 0$ It can be shown that the qualitative similarity between the modes of the exact solution (16) and the solution of (40) breaks down. This breakdown occurs when $\frac{\mu h}{2} = 1$ (this is the so-called Reynolds number, see Morton 1996). On the other hand, the eigenvalues of the de Allen/Southwell/Il'in scheme are always real and are given by

$$\lambda_m = -\frac{\mu}{2h}\coth\frac{\mu h}{2} + \frac{\mu}{h}\sqrt{\coth\mu h - 1}\cos\varphi_m$$

The corresponding eigenvectors are

$$Y_j^m = e^{-\mu x_j/2}\sin\varphi_m$$

(see Lam 1976)

5.2 Fully Discrete Schemes

We now discretise the scheme (38) with respect to time. To this end, we divide the interval $[0,T]$ into N sub-intervals defined by

$$0 = t_0 < t_1 < \ldots < t_N = T\,(k = T/N)$$

We replace the continuous time derivative by divided differences. There are many ways to do this and we only refer to some sources (Stroud 1974, Crouzeix 1975). We concentrate on so-called two-level schemes. To this end, we approximate $\frac{dU}{dt}$ at some time level as follows

$$\frac{dU}{dt} \cong \frac{U^{n+1} - U^n}{k}, \; U^n \equiv U(t_n)$$

For the other terms in (38) we use weighted averages defined as follows:

$$\Phi^{n,\theta} \equiv (1-\theta)\Phi^n + \theta\Phi^{n+1}$$

where $\theta \in [0,1]$. The discrete scheme corresponding to (38) is now defined as follows:

$$\begin{aligned} -\frac{U^{n+1} - U^n}{k} + AU^{n,\theta} &= F \\ U^0 &= U_0 \end{aligned} \tag{42}$$

Some well-known special cases are now given.

$\theta = 0$: the explicit Euler scheme

$$\frac{U^{n+1} - U^n}{k} + AU^n = F \tag{43}$$

$\theta = \frac{1}{2}$: The famous Crank Nicholson scheme

$$\frac{U^{n+1} - U^n}{k} + AU^{n,1/2} = F \tag{44}$$

$\theta = 1$: The fully implicit scheme

$$\frac{U^{n+1} - U^n}{k} + AU^{n+1} = F \tag{45}$$

We are interested in determining if the above schemes are stable (in some sense) and whether their solution converges as $k \to 0$. To this end, we write equation (42) in the equivalent form

$$U^{n+1} = CU^n + H \tag{46}$$

where the matrix C is given by

$$C = (I - kA\theta)^{-1}(I + kA(1 - \theta))$$

and

$$H = -k(I - kA\theta)^{-1}F$$

A well-known result (see Varga 1962) states that the solution of (38) is given by $U(t) = A^{-1}F + exp(tA)\{U(0) - A^{-1}F\}$.

So, in a sense the accuracy of the approximation (46) will be determined by how well the matrix C approximates the exponential function of a matrix. We now discuss this problem.

Definition: Let $A = (a_{ij})$ be an $n \times n$ real matrix with eigenvalues $\lambda_j, j = 1, \dots, n$. The spectral radius $\rho(A)$ is given by

$$\rho(A) = max|\lambda_j|, \;\; j = 1, \dots, n$$

Definition: The time-dependent matrix $T(t)$ is stable for $0 \leq t \leq T$ if $\rho(T(t)) \leq 1$. It is unconditionally stable if $\rho(T(t)) < 1$ for all $0 \leq t \leq \infty$. We now state the main result of this section (see Varga 1962, page 265).

Theorem 10 *Let A be a matrix whose eigenvalues λ_j satisfy $0 < \alpha < Re\,\lambda_j < \beta \;\; \forall\, j = 1, \dots, n$. Then the explicit Euler scheme (43) is stable if*

$$0 \leq k \leq min\left\{\frac{2Re\,\lambda_j}{|\lambda_j|^2}\right\}, \;\; 1 \leq j \leq n \tag{47}$$

while the Crank Nicholson scheme (44) and fully implicit scheme (45) are both unconditionally stable.

Definition: The matrix $T(t)$ is consistent with $exp(-tA)$ if $T(t)$ has a matrix power development about $t = 0$ that agrees through at least linear terms with the expansion of $exp(-tA)$.
The schemes defined by (43), (44) and (45) have matrices that are consistent with the exponential function.

Having determined that the discrete schemes are stable and consistent we are then able to deduce that their solution converges to the exact solution of (38) as $k \to 0$. This is the famous Lax equivalence theorem.

Summarising the steps so far, we approximate parabolic initial boundary value problems in two stages:

Discretising in the spatial direction
Discretising in the time direction

Each discretisation strategy has its own pitfalls. For example, for small values of σ spatial discretisations can introduce spurious oscillations that induce non-physical or 'non-financial' solutions. The time-discretisation may also introduce additional oscillations and these can be propagated in time. Together, spatial and temporal oscillations can lead at best to physically unacceptable solutions and at worst to numerical overflows. These problems have been known for at least 30 years.

5.3 The Fitted Scheme in more detail: Main Results

The previous two sub-sections described semi-discrete and fully discrete schemes, respectively. The approach is vector-based in the sense that the semi-discrete scheme is described by a vector system of ordinary differential equation while the fully-discerete scheme is described by a matrix system. It is possible however, to discretise the system (16)-(18) in both directions simultaneously and this is the approach that we describe here. In particular, we describe an exponentially fitted scheme in the space direction and fully implicit discretisation in the time direction. The results are based on Duffy 1980 where the main theorems are proposed and proved and these results hold for coefficients that depends on both x and t.
We discretise the rectangle $[A, B] \times [0, T]$ as follows:

$$A = x_0 < x_1 < \ldots < x_J = B \;\; (h = x_j - x_{j-1})$$
$$0 = t_0 < t_1 < \ldots < t_N = T \;\; (k = T/N)$$

Consider again the operator L in equation (16) defined by

$$Lu \equiv -\frac{\partial u}{\partial t} + \sigma(x,t)\frac{\partial^2 u}{\partial x^2} + \mu(x,t)\frac{\partial u}{\partial x} + b(x,t)u$$

We replace the derivatives in this operator by their corresponding divided differences and we define the fitted operator L_k^h by

$$L_k^h U_j^n \equiv -\frac{U_j^{n+1} - U_j^n}{k} + \gamma_j^{n+1} D_+ D_- U_j^{n+1} + \mu_j^{n+1} D_0 U_j^{n+1} + b_j^{n+1} U_j^{n+1} \quad (48)$$

Here

$$\varphi_j^{n+1} = \varphi(x_j, t_{n+1}) \text{ in general}$$

and

$$\gamma_j^{n+1} \equiv \frac{\mu_j^{n+1} h}{2} \coth \frac{\mu_j^{n+1} h}{2\sigma_j^{n+1}}$$

Having defined the operator L_h^k we now formulate the fully-discrete scheme that approximates system (16)-(18):

Find a discrete function $\{U_j^n\}$ such that

$$\begin{aligned} & L_k^h U_j^n = f_j^{n+1},\ j = 1, \ldots, J-1,\ n = 0, \ldots, N-1 \\ & U_0^n = g_0(t_n),\ U_J^n = g_1(t_n),\ n = 0, \ldots, N \\ & U_j^0 = \varphi(x_j),\ j = 1, \ldots, J-1 \end{aligned} \quad (49)$$

This is a two-level implicit scheme. We wish to prove that scheme (48) is stable and is consistent with (16)-(18). We prove stability of (48) by the so-called discrete maximum principle instead of the somewhat limited von Neumann stability analysis. The van Neumann approach is well known but the discrete maximum principle is more general and easier to understand and to apply in practice. It is also the de-facto standard technique for proving stability of finite difference and finite element schemes (see Morton 1996, Farrell 2000).

Lemma 1 *Let the discrete function w_j^n satisfy $L_k^h w_j^n \le 0$ in the interior of the mesh with $w_j^n \ge 0$ on the boundary Γ.*
Then $w_j^n \ge 0, \forall j = 0, \ldots, J;\ n = 0, \ldots, N$.

Proof: We transform the inequality $L_k^h \le 0$ into an equivalent vector inequality. To this end, define the vector $U^n = {}^t(U_1^n, \ldots, U_{J-1}^n)$. Then the inequality $L_k^h w_j^n \le 0$ is equivalent to the vector inequality

$$A^n U^{n+1} \ge U^n \quad (50)$$

where

$$\mathbf{A^n} = \begin{pmatrix} \ddots & & \ddots & & 0 \\ & \ddots & & t_j^n & \\ \ddots & & s_j^n & & \ddots \\ & r_j^n & & \ddots & \\ 0 & & \ddots & & \ddots \end{pmatrix}$$

$$r_j^n = \Big(-\frac{\gamma_j^n}{h^2} + \frac{\mu_j^n}{2h}\Big)k$$
$$s_j^n = \Big(\frac{2\gamma_j^n}{h^2} - b_j^n + k^{-1}\Big)k$$
$$t_j^n = \Big(-\Big(\frac{\gamma_j^n}{h^2} + \frac{\mu_j^n}{2h}\Big)\Big)k$$

It is easy to show that the matrix A^n has non-positive off-diagonal elements, has strictly positive diagonal elements and is irreducibly diagonally dominant. Hence (see Varga 1962 pages 84-85) A^n is non-singular and its inverse is positive:

$$(A^n)^{-1} \geq 0$$

Using this result in (50) gives the desired result.

Lemma 2 *Let $\{U_j^n\}$ be the solution of scheme (49) and suppose that*

$$max|U_j^n| \leq m \text{ on } \Gamma$$
$$max|f_j^n| \leq N \text{ in } \Omega$$

Then

$$max_j|U_j^n| \leq -\frac{N}{\beta} + m \text{ in } \bar{Q}$$

Proof: Define the discrete barrier function

$$w_j^n = -\frac{N}{\beta} + m \pm U_j^n$$

Then $w_j^n \geq 0$ on Γ. Furthemore,

$$L_k^h w_j^n \leq 0$$

Hence $w_j^n \geq 0$ in $\bar{Q}$ which proves the result.

Theorem 11 *(Uniform Convergence)*
Let $u(x,t)$ and $\{U_j^n\}$ be the solutions of (16)-(18) and (49), respectively. Then

$$|u(x_j,t_n) - U_j^n| \leq M(h+k) \tag{51}$$

where M is a constant that is independent of h, k and σ.

Remark: This result shows that convergence is assured regardless of the size of σ. No classical scheme (for example, centred differencing in x and Crank Nicholson in time) have error bounds of the form (51) where M is independent of h, k and σ.
Summarising, the advantages of the fitted scheme are:

It is uniformly stable for all values of h, k and σ.
It is oscillation-free. Its solution converges to the exact solution of (16)-(18). In particular, it is a powerful scheme for the Black-Scholes equation and its generalisations.
It is easily programmed, especially if we use real object-oriented design and implementation techniques.

5.4 Graceful Degradation

We now examine some 'extreme' cases in system (49). In particular, we examine the cases

(pure convection/drift) $\sigma \to 0$
(pure diffusion/volatility) $\mu \to 0$

We shall see that the 'limiting' difference schemes are well-known schemes and this is reassuring. To examine the first extreme case we must know what the limiting properties of the hyperbolic cotangent function are:

$$\lim_{\sigma\to 0} \gamma_j^n = \lim_{\sigma\to 0} \frac{\mu_j^n h}{2} \coth \frac{\mu_j^n h}{2\sigma_j^n}$$

We use the formula

$$\lim_{\sigma\to 0} \frac{\mu h}{2} \coth \frac{\mu h}{2\sigma} = \begin{cases} +1 \; if \; \mu > 0 \\ -1 \; if \; \mu < 0 \end{cases}$$

Inserting this result into the first equation in (49) gives us the first-order scheme

$$\mu > 0, \quad -\frac{U_j^{n+1} - U_j^n}{k} + \mu\frac{(U_{j+1}^{n+1} - U_j^{n+1})}{h} + b_j^{n+1} U_j^{n+1} = f_j^{n+1}$$
$$\mu < 0, \quad -\frac{U_j^{n+1} - U_j^n}{k} + \mu\frac{(U_j^{n+1} - U_{j-1}^{n+1})}{h} + b_j^{n+1} U_j^{n+1} = f_j^{n+1}$$

These are so-called implicit upwind schemes and are stable and convergent (Duffy 1977, Dautray 1993). We thus conclude that the fitted scheme degrades to an acceptable scheme in the limit. The case $\mu \to 0$ uses the formula

$$\lim_{x\to 0} x \coth x = 1$$

Then the first equation in system(49) reduces to the equation

$$-\frac{U_j^{n+1} - U_j^n}{k} + \sigma_j^n D_+ D_- U_j^{n+1} + b_j^n U_j^{n+1} = f_j^{n+1}$$

This is a standard approximation to pure diffusion problems and such schemes can be found in standard numerical analysis textbooks (see for example, Press 1989).
These limiting cases reassure us that the fitted method behaves well for 'extreme' parameters values.

5.5 An explicit Time-Marching Scheme

It is interesting to investigate the use of fitting in combination with explicit time marching. We do not expect the corresponding scheme to be unconditionally stable. The scheme is:

$$-\frac{U_j^{n+1} - U_j^n}{k} + \gamma_j^n D_+ D_- U_j^n + \mu_j^n D_0 U_j^n + b_j^n U_j^n = f_j^n \qquad (52)$$

Rearranging terms in (52) gives

$$U_j^{n+1} = A_j^n U_{j+1}^n + B_j^n U_j^n + C_j^n U_{j-1}^n \tag{53}$$

where

$$A_j^n = \frac{\gamma_j^n}{h^2} + \frac{\mu_j^n}{2h}, \quad B_j^n = k^{-1} - \frac{2\gamma_j^n}{h^2}, \quad C_j^n = \frac{\gamma_j^n}{h^2} - \frac{\mu_j^n}{2h}$$

If each of the coefficients A_j^n, B_j^n and C_j^n are non-negative then the right-hand side of (53) will be positive, thus leading us to the conclusion that $U_j^{n+1} \geq 0$. In this case $B_j^n \geq 0$ if

$$k^{-1} - \frac{2\gamma_j^n}{h^2} \geq 0$$

or

$$\frac{\mu_j^n k}{h} \leq \tanh \frac{\mu_j^n h}{2\sigma_j^n} \tag{54}$$

Inequality (54) is a variation of the famous Courant-Friedrichs-Lewy (CFL) condition. If we let $\sigma \to 0$ in (54) the limiting case of (54) becomes

$$\frac{|\mu_j^n| k}{h} \leq 1$$

which is precisely the CFL condition for first-order hyperbolic equations! The corresponding reduced scheme is called the explicit upwind scheme (see Dautray 1993, page 99):

$$\mu > 0, \quad -\frac{U_j^{n+1} - U_j^n}{k} + \mu\Big(\frac{U_{j+1}^n - U_j^n}{h}\Big) = 0$$
$$\mu < 0, \quad -\frac{U_j^{n+1} - U_j^n}{k} + \mu\Big(\frac{U_j^n - U_{j-1}^n}{h}\Big) = 0$$

Again, these are reassuring results.

6 Approximating the Greeks

We now investigate how to approximate the derivatives of the solution of the system (16)-(18). These quantities are called the sensitivities or 'greeks' and the most important ones are:

Delta Δ
Gamma Γ

These quantities are needed when we wish to reduce the sensitivities of a portfolio or an option to the movements of the underlying asset.
In general, the payoff function is continuous but its derivatives are discontinuous. We suspect that divided difference approximations to Δ and Γ may give problems. In particular, we suspect that things go wrong when the stock price

is close to the exercise price ('at the money'). In fact, numerical experiments show that the Crank Nicholson scheme gives oscillating solutions for the delta and gamma when the option is at the money while no such oscillations occur with the fitted schemes. These were calculated by the following formulas:

$$\begin{aligned} \Delta &\approx (c_{j+1}^n - c_{j-1}^n)/2h, \quad h = S_{j+1} - S_j \\ \Gamma &\approx (c_{j+1}^n - 2c_j^n + c_{j-1}^n)/h^2 \end{aligned} \tag{55}$$

where c_j^n denotes the value of the option price at grid point S_j and at time level t_n.

We know that Δ and Γ oscillate in the case of the Crank Nicholson scheme for smaller values of the volatility or for large values of the interest rate. The Duffy fitted scheme approximates Δ well although we see some 'flattening' and dispersion in the approximation to Γ. The problem of approximating the greeks in the case of mixed or even Neumann boundary conditions becomes even more challenging that in the case of Dirichlet boundary conditions because we must approximate one-sided derivatives at the boundaries (see Topper 2000 for a discussion of this problem). The most general case is:

$$-\frac{\partial c}{\partial t} + \sigma \frac{\partial^2 c}{\partial S^2} + \mu \frac{\partial c}{\partial S} + bc = f, \quad (S,t) \in (A,B) \times (0,T) \tag{56}$$

$$\begin{aligned} &u(S,0) = g(S), \quad S \in (A,B) \\ &\alpha_0 c(A,t) + \alpha_1 \sigma \frac{\partial c}{\partial S}(B,t) = g_0(t) \\ &\beta_0 c(B,t) + \beta_1 \sigma \frac{\partial c}{\partial S}(B,t) = g_1(t) \end{aligned} \tag{57}$$

It is possible to approximate (56) by standard finite difference schemes (see Wilmott 1993) but instead of discussing this approach we write (56) as a system of first-order equations as follows:

$$\begin{aligned} &-\sigma \frac{\partial c}{\partial t} + \sigma \frac{\partial V}{\partial S} + \mu V + \sigma bc = \sigma f \\ &\sigma \frac{\partial c}{\partial S} = V \\ &c(S,0) = g(S) \\ &\alpha_0 c(A,t) + \alpha_1 V(A,t) = g_0(t) \\ &\beta_0 c(B,t) + \beta_1 V(B,t) = g_1(t) \end{aligned} \tag{58}$$

Our aim is to approximate both c and V and having done that we will have values for both the option price and its delta! Furthermore, we do not have to worry about approximating the derivative quantities at the boundary conditions. A robust and simple scheme for approximating (58) was proposed by Herbert Keller (see Keller 1970) and it is based on a so-called control element in which

the derivatives with respect to S and t are approximated.
Let us assume (for convenience only!) that σ and μ are constant, that $f \equiv 0,\ b \equiv 0$ and that $\alpha_0 = \beta_0 = 1,\ \alpha_1 = \beta_1 = 0$ (the most general case is discussed in Keller 1971). Define the quantities

$$S^n_{j\pm 1/2} = \frac{1}{2}(S^n_j + S^n_{j\pm 1})$$

and

$$\begin{aligned}
t_{n\pm 1/2} &= \frac{1}{2}(t_n + t_{n\pm 1}) \\
c^n_{j\pm 1/2} &= \frac{1}{2}(c^n_j + c^n_{j\pm 1}) \\
c^{n\pm 1/2}_j &= \frac{1}{2}(c^n_j + c^{n\pm 1}_j) \\
D^+_x c^n_j &= h^{-1}(c^n_{j+1} - c^n_j) \\
D^+_t c^n_j &= k^{-1}(c^{n+1}_j - c^n_j)
\end{aligned}$$

The so-called Keller box scheme is defined as

$$\begin{aligned}
&-\sigma D^+_t c^n_{j+1/2} + \sigma D^+_x V^{n+1/2}_j + \mu V^{n+1/2}_{j+1/2} = 0 \\
&\sigma D^+_x c^n_j = V^n_{j+1/2} \\
&\left.\begin{array}{l} u_0(t) = g_0(t) \\ u_J(t) = g_1(t) \end{array}\right\} j = 1, \ldots, J-1
\end{aligned} \tag{59}$$

The advantages of this scheme have been documented in Keller 1970 and Lam 1976. The main ones are:

- it has second order accuracy
- it is unconditionally stable
- the data and coefficients need only be piecewise smooth (as is the case in option modelling)
- it is A-stable (that is, if the exact solution decays in time then so does the numerical solution)
- it is easy to program

The motivation for (59) can be given by discretising (58) in the space direction first, by centred differencing and then in t. We finish this section with a discussion of the semi-discrete scheme for (58). This is given by

$$-\sigma \frac{dc_{j+1/2}}{dt} + \sigma D_+ V_j + \mu V_{j+1/2} = 0, \ \ j = 0, \ldots, J-1 \tag{60}$$

$$\sigma D^+_x c^n_j = V^n_{j+1/2}$$

$$c_j(0) = g(S_j), \ \ j = 0, \ldots, J \tag{61}$$

$$\left.\begin{array}{l} c_0(t) = g_0(t) \\ c_J(t) = g_1(t) \end{array}\right\} j = 1, \ldots, J-1 \tag{62}$$

We now combine the terms in (60) to give a system of ODEs which will be shown to be A-stable. To this end, we need some preliminary results whose proofs are easy.

Lemma 3

$$\left.\begin{array}{ll} (a) & D_0 V_j = \sigma D_+ D_- c_j \\ (b) & V_{j+1/2} + V_{j-1/2} = 2\sigma D_0 c_j \\ (c) & D_+ V_j + D_+ V_{j-1} = 2 D_0 V_j \end{array}\right\} j = 1, \ldots, J-1$$

We now write (60) at two consecutive mesh points S_j and S_{j-1} and add the results to give

$$-\sigma \frac{d}{dt}\{c_{j+1/2} + c_{j-1/2}\} + \sigma\{D_+ V_j + D_+ V_{j-1}\} + \mu\{V_{j+1/2} + V_{j-1/2}\} = 0$$

Using the above lemma we see that the above equation can be written as

$$\frac{1}{4}\frac{d}{dt}(c_{j-1} + 2c_j + c_{j+1}) + \sigma D + D_- c_j + \mu D_0 c_j = 0$$

Finally, this equation is written as a system of ODEs

$$C\frac{dU}{dt} + AU = 0, \quad U(0) = U_0 \tag{63}$$

where

$$\mathbf{C} = \begin{pmatrix} 2 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 2 \end{pmatrix}$$

and the matrix A is already known. If we use fitting in (63) we see that the system is unconditionally stable. Similar conclusions can be found in Lam 1976. The inverse of the matrix C is called the mass matrix and it occurs in many finite element approximations to boundary value problems. We can view (63) as a 'weighted' version of (38).

To conclude this section, we see the advantages of the box scheme for risk as follows:

- it has second order accuracy in both the option price value and its
- it can handle discontinuous coefficients (e.g. when the option is at the money), a situation that often arises in valuation theory
- it can handle Neumann and Robins type boundary conditions
- it resolves many of the oscillation problems that are experienced with the Crank Nicholson method (see Lam 1976)

7 Numerical Results and Coding

We now discuss how the fitted and classical numerical schemes in this article are implemented.To this end, we have programmed solutions in C++. This is an object-oriented language and it is a de facto standard in many branches of finance. But before we are able to write C++ code we must transform the numerical schemes into lower-level entities. In particular, there are a number of requirements:
Reliability: we need a guarantee that the schemes are mapped accurately to C++ and that no logic errors are made during the mapping. This is achieved by modelling entities (for example, a partial differential equation or an ODE as a C++ class). Documentation is based on the Unified Modeling Language UML) (see Booch 1999).
Flexibility: All parameters, boundary conditions, matrix solvers and so on can be configured at run-time or start-up time. Thus, there is as little hard wiring as possible in the code. This level of flexibility is achieved by the use of the popular design patterns (see Gamma 1994).
Understandability: It must be possible to adapt and extend the code in a matter of days rather than weeks or months. We used 'flat' C code in the past but this approach was not satisfactory because of the length of time it took to get a program up and running and the problem of code comprehension due to the large 'cognitive gap' between problem space (in this case two-point initial boundary value problems) and the solution space (C code).

For output, we have use the popular spreadsheet Excel for two-dimensional representation and OpenGL for three-dimensional representation. In order to test and compare our schemes, we looked at the following scenarios:

Scenario 1: Calculating the option price using the exact solution, Crank Nicholson and fitted schemes for 'normal' values of the volatility and interest rate

———fig A———-

Here we take $T = 1, \sigma = 0.2, K = 100, r = 0.06$.
Here we see that both Crank Nicholson and fitted methods approximate the exact solution well.

However, small oscillations do exist in the Crank Nicholson scheme although they are not visible for small and medium values of the underlying S. It can be shown experimentally that the approximate solution from CN oscillates at large values of S. The solution oscillates wildly. This phenomenon has also been experienced with CN when applied to fluid mechanics problems. What is the reason for this oscillatory behaviour? The reason is that CN is only weakly stable (see Peaceman 1977, chapter 4) and this is closely related to so-called numeric dispersion, in particular numerical dispersion acts to stabilize the difference equation. In general, positive dispersion is associated with stability while

negative dispersion is associated with instability. The CN scheme has zero numerical dispersion!
We note that the classical fully implicit scheme (no fitting), that is discrete scheme (45) has good dispersion properties. *Scenario 2*: Calculating the option price using the exact solution, Crank Nicholson and fitted schemes for 'extreme' values of the volatility and interest rate. In particular, we investigate the accuracy when the volatility is small and the interest rate is large

—— fig B——

The coefficients are the same as with scenario 1 except $\sigma = 0.001$.
Here we see that the Crank Nicholson oscillates for large values of the underlying. The oscillations are present for smaller values of the underlying but they are magnified. This is because the Crank Nicholson method is neutrally stable.

Scenario 3: Calculating the values of the greeks using the exact solution, Crank Nicholson and fitted schemes for 'normal' values of the volatility and interest rate

——— fig C———

Here we see that both Crank Nicholson and fitted methods approximate the delta and gamma well.

Scenario 4: Calculating the values of the greeks using the exact solution, Crank Nicholson and fitted schemes for 'extreme' values of the volatility and interest rate. In particular, we investigate the accuracy when the volatility is small and the interest rate is large.

———fig D———

Here we see that the Crank Nichsolson method produces bounded oscillations solutions.
The spurious oscillations that arise in CN are pronounced when we approximate delta and gamma. The oscillations occur when the option is 'at the money' and this strange behaviour is not well documented in the literature devoted to the numerical solution of the Black Scholes equation. Another way to approximate the option price and its delta to second-order accuracy is to use the Keller box scheme (59). This scheme can also handle discontinuous coefficients (for example, the derivative of the payback function when S = K).

Scenarion 5:Our results compare favourably with binomial and trinomial methods and in fact outperform them in a number of respects. First, binomial methods consume a lot of computer memory because a lattice needs to be built for each specific price of S. Furthermore, it is well known that the values from this method oscillate in a 'yo-yo' manner unless a large number of time intervals is given. The trinomial method suffers from the same ills and is in fact equiva-

lent to an explicit finite difference scheme, which by definition is conditionally stable. Thus, there is a dependency between the physical parameters (such as the volatility and drift) and the required number of time intervals. We prefer implicit difference schemes. Our conclusion is that the trinomial method should be complemented by robust difference schemes. They perform at least as well as the trinomial method and are based on the work of mathematicians from the last three centuries.
Some theoretical disadvantages and shortcomings of the binomial and trinomial methods in the author's opinion are: 1) most methods (e.g. CRR, JR (see Leisen 1996) are only first-order convergent. This means that convergence is linear at best 2) many authors use the root-mean-square (RMS) as the test of convergence even though it can be proved that such Euclidean norms are not trustworthy in general (see Farrell 2000 for a counterexample). It is possible to achieve high-order convergence by applying multi-step Runge-Kutta methods to the semi-discrete scheme (38) 4) the binomial and trinomial assume that the behaviour of the underlying S obeys some given distribution type. The generalized Black-Scholes equation and its finite difference approximations do not use such assumptions (at least not explicitly).

8 Conclusions

We have presented a new finite difference scheme that approximates one-factor Black-Scholes type equations. We have discussed its stability and convergence properties in detail and numerical experiments have shown that it can compete well with classical difference schemes. We summarise its main advantages as follows:

Functionality and *Accuracy*: It can be applied to a wide range of non-constant coefficient parabolic equations. We have shown that the method works for Dirichlet boundary and the fitting factor technique can be applied to problems with mixed boundary conditions by rewriting the original equations as a first-order system and then applying a 'fitted' box scheme. Higher order accuracy will be achieved by using Richardson extrapolation. The method produces no spurious oscillations even in the limiting cases.
Performance: The method performs well but not always as well as the Crank Nicholson method. This is due to the fact that the hyperbolic cotangent function needs to be calculated at each mesh point and time-level. We prefer the finite difference method to finite element methods because it is easy to program. Our opinion is that the finite element method is overkill for the current crop of partial differential equations that model option behaviour (see Zvan 1998). It has been shown that classical finite elements behave just as badly as classical finite difference schemes when the volatility is small or when the drift is large (see Ikeda 1983, Farrell 2000). However, classical methods perform well when the parameters are not too small or too large and fitted methods may be too expensive in such cases.However, if you want robustness you should use fitted

methods.
Reliability: The fitted method is extremely robust and we have proved both mathematically and numerically that it always produces good results. It is more reliable than Crank Nicholson. It is a vast improvement on more traditional methods (such as the binomial and trinomial methods), both numerically and mathematically.
Portability: The fitted method has been applied to two-dimensional steady-state problems (see Farrell 2000) and the author is now involved in applying the method to two-factor time-dependent problems (for example, Asian options and bond pricing) using 'direct', Alternating Direction Implicit (ADI) and splitting methods (see Yanenko 1971). These methods can be applied to multi -asset modelling, and path-dependent problems.

Acknowledgements

A number of people and organizations helped in the preparation of this article, both directly and indirectly. First of all, Enterprise Ireland and Datasim Education Ltd. Dublin Ireland who sponsored part of the research and implementation work. In particular, Isabel Casas and Michael Cooney developed much of the C and C++ code. Special thanks are due to Alexander Levin (Dime Savings Bank) and Peter Carr (Banc of America) who helped with some pricing models. Finally, I thank Jasmin Turkic who programmed the binomial and trinomial methods in C++ and developed drivers for Excel and OpenGL.

References

[1] de Allen, D., R. Southwell Relaxation Methods applied to determining the motion, in two dimensions, of a viscous fluid past a fixed cylinder (1955). Quart. J. Mech. Appl. Math. 8, 129-145.

[2] Barraquand, J., T. Pudet Pricing of American Path-dependent contingent Claims (1996). Mathematical Finance, volume 6, No. 1.

[3] Bhansali, V. Exotic and Hybrid Options (1998). McGraw-Hill New York. Bhar, R. et al The Reduction of forward Rate dependent Volatility HJM models to Markovian form: pricing European Bond Options (2000). Journal of Computational Finance Volume 3/Number 3 Spring 2000.

[4] Black, F., M. Scholes The pricing of options and corporate liabilities (1973). Journal of Political Economy 81, 637-659.

[5] Booch, G, et al The Unified Modeling Language User Manual (1999) Addison-Wesley Reading MA

[6] Boyle. P. Options: A Monte Carlo Approach (1977) Journal of Financial Economics 4 323-338.

[7] Carr, Peter, D. B. Madan Option Valuation Using the Fast Fourier Transform (1998). Report Morgan Stanley New York.

[8] Cox, J. C., M. Rubinstein Options Markets (1985) Prentice-Hall Englewood Cliffs New Jersey.

[9] Richtmeyer, R. and K. W. Morton Difference Methods for Initial-Value Problems (1967) Interscience New York

[10] Crouzeix, M. On the Approximation of linear Operational Differential Equations by the Runge-Kutta Method (1975), Doctorate Thesis, Paris VI University, France.

[11] Dautray, R., J. L. Lions Mathematical Analysis and Numerical Methods for Science and Technology, Volume 6 (1993). Springer-Verlag Berlin.

[12] Dennis, S. C. R., J. D. Hudson Further Accurate Representations of Partial Differential Equations by Finite-difference Methods. J. Inst. Maths Applics 26, 369-379.

[13] Van Deventer, D. R., K. Imai Financial Risk Analytics (1997) McGraw-Hill Chicago.

[14] Doolan, E. P. et al Uniform Numerical Methods for Problems with Initial and Boundary Layers (1980) Boole Press, Dublin, Ireland.

[15] Duffy, D. Finite Elements for mixed Initial Boundary Value Problems for Hyperbolic Systems of Dissipative Type (1977), M.Sc. Thesis Trinity College, Dublin Ireland.

[16] Duffy, D. Uniformly Convergent Difference Schemes for Problems with a small Parameter in the leading Derivative (1980), Ph.D. Thesis Trinity College, Dublin Ireland.

[17] Emel'yanov, K. V. A Difference Method for solving the third Boundary Value Problem for a Differential Equation with a small Parameter in its leading Derivative (1975). Zh. Vychisl. Mat. mat. Fiz. 15, 6, 1466-1481.

[18] Emel'yanov, K. V. A Difference Scheme for an Ordinary Differential Equation with a small Parameter (1978). Zh. Vychisl. Mat. mat. Fiz. 18, 5, 1146-1153.

[19] Farrell, P. et al Robust Computational Techniques for Boundary Layers (2000). Chapman and Hall/ CRC Boca Raton.

[20] Friedrichs, K. O. Symmetric Positive Linear Differential Equations (1958) Communications on Pure and Applied Mathematics, Vol. XI, 333-418.

[21] Gamma, E. et al Design Patterns: Elements of Reusable Object-Oriented Software (1994) Addison-Wesley Reading MA.

[22] Gemmil, G. Options Pricing (1993) McGraw-Hill London U.K.

[23] Godounov, S. Equations of Mathematical Physics (1973) (in French). Editions Mir, Moscow.

[24] Godounov, S.et al Numerical Resolution of Multidimensional Problems in Gas Dynamics (1979) (in French). Editions Mir, Moscow.

[25] Gushchin, V. A., V. V. Schennikov A Monotonic Difference Scheme of Second-order Accuracy (1973). Zh. Vychisl. Mat. mat. Fiz. 14, 3, 789-792.

[26] Hull, J. Options, Futures and other Derivative Securities (1993) (second edition) Prentice-Hall International Englewood Cliffs, New Jersey.

[27] Ikeda, T. Maximum Principle in Finite Element Models for Convection-Diffusion Phenomena (1983). North-Holland Amsterdam.

[28] Il'in, A.M. Differencing Scheme for a differential equation with a small parameter affecting the highest derivative (1969). Mat. Zametki 6, 237-248.

[29] Il'in, A. M., L. A. Lyusternik Linear equations of the second order of parabolic type. Russian Maths. Surveys.

[30] Keller, Herbert A new Difference Scheme for Parabolic Problems (1970). Numerical Solution of Partial Differential Equations-II, Synspade 1970 (B. Hubbard, ed.)

[31] Lam, D. C. L., R. B. Simpson Centered Differencing and the Box Scheme for Diffusion Convection Problems (1976). Journal of Computational Physics 22, 486-500

[32] Levin, Alexander Two-Factor Gaussian Term Structure: Analytics, Historical Fit and Stable Finite-Difference Pricing Schemes, Lecture held at Courant Institute, New York University May 2000.

[33] Leisen, D., M. Reimer Binomial Models for Option Valuation - examining and improving Convergence (1996). Applied Mathematcal Finance 3, 319-346.

[34] Morton, K., Numerical Solution of Convection-Diffusion Equations (1996). Chapman and Hall, London UK.

[35] Peaceman, D. (1977) Fundamentals of Numerical Reservoir Simulation. Elsevier Scientific Publishing, Amsterdam.

[36] Press, W. H. et al Numerical Recipes (Fortran Version) (1989) Cambridge University Press U.K.

[37] Roscoe, D. F. New Methods for the Derivation of Stable Difference Representations (1975). J. Inst. Maths. Applics 16, 291-301.

[38] Rogers, L. C. G., Z. Shi The Value of an Asian Option (1995). J. Appl. Prob. 32, 1077-1088.

[39] Samarski, A. A. Some Questions from the General Theory of Difference Schemes (1976). Amer. Math. Soc. Trans. (2), Vol. 105.

[40] Smith, G. D. Smith Numerical Solution of Partial Differential Equations: Finite Difference Methods (1978). Oxford University Press.

[41] Stoyan, Gisbert Monotone Difference Schemes for Diffusion-Convection Problems (1979a). ZAMM 59, 361-372.

[42] Stoyan, Gisbert On a Maximum Principle for Matrices, and on Conservation of Monotonicity with Applications to Discretisation Methods (1979b) ZAMM 62, 375-381.

[43] Stoyan, Gisbert Towards a general-purpose difference scheme for the linear one-dimensional parabolic equation (1982) Proceedings of the Seventh International Summer School, Berlin, Akademie-Verlag Berlin 1981.

[44] Stroud, A. H. Numerical Quadrature and Solution of Ordinary Differential Equations (1974) Springer-Verlag New York.

[45] Tavella, D., C. Randall Pricing Financial Instruments: The Finite Difference Method (2000) John Wiley and Sons New York.

[46] Topper, Juergen. Finite Element Modeling of Exotic Options (2000). Operations Research Proceedings, Springer Verlag Berlin.

[47] Varga, R. Matrix Iterative Analysis (1962). Prentice-Hall Inc. Englewood Cliffs NJ.

[48] Vishik, V. I., L. A. Lyusternik Regular degeneration and boundary layer for linear differential equations with a small parameter (1957). Usp. Math. Nauk. 12, pages 13-122.

[49] Yanenko The Method of Fractional Steps (1971). Springer Verlag New York.

[50] Zvan, R. et al Robust numerical Methods for PDE Models of Asian Options (1997). Journal of Computational Finance Volume 1/Number 2, Winter 1997/1998.

[51] Zvan, R., P. Forsyth, K. Vetzal (1998) A general finite element approach for PDE option pricing models. Working paper, University of Waterloo.

[52] Wilmott, P., J. Dewynne, S. Howison (1993) Option Pricing. Oxford Financial Press, Oxford UK.