Numerical Analysis of Jump Diffusion Models: A Partial Differential Equation Approach

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Summary

We discuss a number of numerical methods that approximate the solution of the Partial Integro Differential Equation (PIDE) that models contingent claims with jumps. Starting with the Merton jump model for the underlying asset we motivate how to find the corresponding PIDE that models a derivative quantity on that asset. We pose the PIDE in a form that becomes amenable to a solution using the finite difference method (FDM). We discuss a number of schemes and we focus on one-factor models and we concentrate on improving the Black Scholes for European options. The models can be extended to multi-factor option pricing problems but a discussion is outside the scope of this article. The main goal of this article is to show the relationship between jump models, PIDEs and numerical analysis. We see this synergy as a new development in computational finance and new techniques need to be developed and elaborated upon in the coming years.

Background to Jump Diffusion Models

It has been known for some time that Brownian motion does not model well the behaviour of stocks (Heston 1993, Bates 1991, Mandelbrot 2004). A number of extensions have been proposed. For example, Heston has produced a closed form solution for option models with stochastic volatility and the method has been extended to model problems where the underlying follows discontinuous stochastic processes. In particular, we can use the Fourier inversion technique to model problems involving Levy processes (Carr 2002).

The original Black Scholes model and its extensions assume that the probability distribution of the stock price at any given future time is lognormal. If this assumption is not true we shall get biases in the prices produced by the model.

If the true distribution is different from the lognormal distribution we shall underprice or overprice call and put options, depending on the distributions' tails (Hull 2000). A number of models have been proposed in order to resolve these shortcomings, for example:

- Model the volatility as a stochastic process (for example, the Heston model)
- Models where the company's equity is assumed to be an option on its asset
- Models where the stock price may experiences occasional jumps rather than continuous changes as happened on 19 and 20 October 1987, for example (Bates 1991). New models must thus provide some support for random jumps

We now discuss how to incorporate jump models into a PIDE formulation. We note that very little has been done on this area to date although a number of initial results exist.

Formulation of the Continuous Problem

There is evidence to suggest that the well-loved geometric Brownian motion model for stock price behaviour does not always work well in practice. In particular, financial instruments do not follow a lognormal random walk (see for example, Bates 1991, Wilmott 1998). Jumps can appear at random times and to this end a number of alternative models have been proposed, for example the jump diffusion model (see Merton 1976). A particular case is the Poisson process. The Poisson process is a special case of a so-called counting process. In general, a random process $X(t)$ is said to be a counting process if $X(t)$ represents the total number of events that have occurred in the time interval $(0, t)$.
A counting process must satisfy the following conditions:

1. $X(t) \geq 0, X(0) = 0$
2. $X(t)$ is integer-valued
3. $X(s) < X(t)$ if $s < t$
4. $X(t) - X(s)$ equals the number of events that have occurred in the interval (s, t)

A Poisson process X(t) is a counting process with rate or intensity $\lambda > 0$ if

1. $X(0) = 0$
2. $X(t)$ has independent and stationary increments
3. $P[X(t + dt) - X(t) = 1] = \lambda dt + o(dt)$
4. $P[X(t + dt) - X(t) \geq 2] = o(dt)$

when $o(dt)$ is a function that tends to zero faster then dt, that is $\lim_{dt \to o} \frac{o(dt)}{dt} = 0$ (Hsu 1997)

In the current context we prefer to define a Poisson process dq as follows:

$$dq = \begin{cases} 0, \text{with probability } 1 - \lambda \, d \, t \\ 1, \text{with probability } \lambda \, d \, t \end{cases} \tag{1}$$

where $\lambda =$ Poisson arrival intensity

Thus, there is a probability λdt of a jump in the timestep dt. Actually, the Poisson process models many kinds of arrival patterns and its applications are numerous, for example queuing systems, inventory control applications and telecommunications. It also models the behaviour of underlyings in real options modelling:

- Energy prices (Pilipović 1998)
- Oil prices
- Business models (Mun 2002)

These quantities can exhibit peaks and spikes in their respective prices; for example, in one case the unit price of natural gas in the Netherlands jumped from 30 Euro to more than 1500 Euro in one day during a period of short supply. Fortunately, the price dropped again shortly afterwards. With shares however, the price plummets as was witnessed in October 1987. The modified stochastic differential equation for an underlying with jumps is:

$$
\begin{aligned}
&\frac{dS}{S} \quad = \mu d\,t + \sigma\, dz + (\eta - 1)\, d\,q \\
&\text{where} \\
&S \quad = \text{Underlying stock price} \\
&\mu \quad = \text{Drift rate} \\
&\sigma \quad = \text{Volatility} \\
&dz \quad = \text{Increment of Gauss-Wiener process} \\
&dq \quad = \text{Poisson process} \\
&\eta - 1 \; = \text{Impulse function producing a jump from } S \text{ to } S\eta \\
&K \quad = E(\eta - 1), \text{expected relative jump size}
\end{aligned}
\tag{2}
$$

In other words, the arrival of a jump is random and this is part of the stochastic differential equation for S. We thus have two sources of uncertainty. First, the term σdz corresponds to the usual Brownian motion while the term dq corresponds to exceptional (and infrequent) events. Two special cases of equation (2) are geometric Brownian motion and the case of pure jump diffusion

$$\frac{dS}{S} = \eta - 1$$

In this case the path followed by S is continuous most of the time while finite negative or positive jumps will appear at discrete points in time. We assume that Brownian motion and jumps are independent. Based on the SDE (2) the resulting PIDE for a contingent claim V(S,t) that depends on S is given by (Briani 2004, Merton 1976):

$$
\begin{aligned}
&\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda K) S \frac{\partial V}{\partial S} - rV + (\lambda \int_0^\infty V(S\eta) g(\eta) d\eta - \lambda V) \\
&\text{where} \\
&\tau \; = \; T - t = \text{time to expiry} \\
&\eta \; = \text{jump amplitude} \\
&\text{and} \\
&g(\eta) \geq 0, \int_0^\infty g(\eta) d\eta = 1
\end{aligned}
\tag{3}
$$

We rewrite this equation in the more tractable form:

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (\tau - \lambda K) S \frac{\partial V}{\partial S} - (r + \lambda) V$$
$$+\lambda \int_0^\infty V(S\eta) g(\eta) d\eta \tag{4}$$

We must augment this PIDE by defining the usual payoff terminal conditions and the boundary conditions when S is zero and infinity. One choice is the linearity boundary conditions at both end points, namely the second-order derivatives with respect to S are zero (see Duffy 2004 on the numerical approximation).

Moving to the Numerical Analysis: Some Challenges

In general, an exact or closed solution to the problem (4) is impossible in general, due mainly to the integral term. For this reason, we must approximate it using approximate methods. In this article we concentrate on the application of the finite difference method. We reduce the scope even further by focusing on one-factor models. Two-factor and three-factor PIDE models are also being studied in the financial literature but a discussion of these is outside the scope of this article. Looking at equation (4) we see that it consists of a 'classical' Black Scholes part (convection-diffusion equation) and an integral part (that corresponds to the 'memory' in the system as it were and this is the term that models jumps). It is intuitively obvious that the PIDE (4) will be harder to solve numerically than the corresponding Black Scholes PDE. To be more precise, we list a number of hurdles to be crossed:

- The integral term in the PIDE is on a semi-infinite interval. How are we going to approximate the integral term? In some cases the integrand may also have singularities. In general, we speak of the non-local nature of the integral term
- The PDE part of the PIDE is also defined on an infinite interval. It too must be truncated, but how? Furthermore, we are now confronted by two truncated intervals and how should they be chosen so as not to destroy accuracy?
- Problem (4) may not have a solution in the classical sense and we then may need to resort to so-called viscosity solutions (Briani 2004, Crandall 1992). For example, the solution of (4) may not be continuously differentiable and we may have to find weaker solutions

- Can we apply standard finite difference schemes (Euler, Crank Nicolson, fitting) in combination with numerical integration techniques to produce stable and accurate approximations? We may still experience spurious oscillation problems that Crank Nicolson is famous for (Duffy 2004a, Duffy 1980)
- How can we avoid producing a dense system of equations when we approximate the integral term on a bounded interval?
- How to compare one method with another one? For example, is Crank Nicolson really better than implicit Euler even though the former method produces spurious oscillations?
- Finally, having truncated the domain of integration, how do we define boundary conditions at the boundaries of the truncated domain? (Briani 2004)

We wish to address the above issues in this article.

Numerical Analysis of PIDE for Jump Processes and PIDE

We now give an overview of a number of finite difference schemes that approximate the solution of (4). In fact, we must approximate the solutions of the following initial boundary value problem associated with (4):

$$\frac{\partial u}{\partial t} = Lu + \lambda \int_{-\infty}^{\infty} u(x+y,t)g(y)dy \tag{5}$$

where L is the convection-diffusion part of the PIDE. This PIDE is written in a slightly different form than (4). Since we are using the "engineer's" time we need to augment the PIDE by an initial condition:

$$u(x,0) = \psi(x) \text{ for } -\infty < x < \infty \tag{6}$$

The corresponding boundary conditions are always an issue in these kinds of problems and we use the following ones (there are other alternatives but they are not the most pressing issues at this stage):

$$\frac{\partial^2 u}{\partial x^2} = 0 \text{ as } x \to \infty \text{ and } x \to -\infty \tag{7}$$

The situation is complicated by the fact that the unknown solution appears in

the differential equation and the integral term. In general, we then must construct two meshes, namely one for the PDE and one for the integral term. These meshes do not necessarily have to coincide but things become messy in this case because we have to use some kind of interpolation when we construct the discrete systems of equations. It is easier to use the same mesh for both the differential and integral terms.

Another potential problem is that the discrete system of equations can result in a dense matrix. In the pure PDE we get a band matrix of some kind (for example, a tridiagonal system) but again the integral term confuses things. We shall see how to avoid this problem.

Finally, the PDE is defined on a semi-infinite interval and we must truncate this to a finite interval. Furthermore, the integral term in equation (5) is on an infinite interval and this must be truncated. This issue is discussed in La Chioma 2003 and we give the main results here. In general, the procedure is to choose two finite values A and B such that the difference between the infinite and truncated integrals is less than a given tolerance:

$$|\int_{-\infty}^{\infty} f(x)dx - \int_{A}^{B} f(x)dx \,| < \epsilon$$

We approximate the truncated integral by some kind of Newton-Cotes integration method:

$$\int_{A}^{B} f(x)dx \approx \frac{B-A}{N}\sum_{j=0}^{N} w_j f(x_j)$$

In the current problem we have a specific integrand (kernel), namely a probability density function of the form

$$\Gamma_\delta(y) = \tfrac{1}{\sqrt{2\pi}\delta}\exp(-\tfrac{y^2}{2\delta^2})$$

This function goes to zero very quickly and we only look at this when:

$$\Gamma_\delta(y) \geq \epsilon \Leftrightarrow$$

$$-\sqrt{-2\,\delta^2 \log(\epsilon\sqrt{2\pi})} \leq y \leq \sqrt{-2\,\delta^2 \log(\epsilon\,\delta\sqrt{2\pi})}$$

In the above context we then choose the limits of integration as follows:

$$\begin{cases} A = +\sqrt{-2\,\delta^2 \log(\epsilon\delta\sqrt{2\pi})} \\ B = -A \end{cases}$$

We now propose the modified form of equation (5)

$$\frac{\partial u}{\partial t} = Lu + \lambda \int_A^B u(x+y,t)\Gamma_\delta(y)dy \tag{8}$$

We have now replaced the integral on an infinite interval by one on a finite interval. Furthermore, we have also truncated the domain corresponding to the PDE part of the initial boundary value problem. We are now ready to look at candidate finite difference solutions.

Comparison of the different Numerical Methods

In the last few years there has been a rapid growth in the interest in PIDE for financial derivatives. We discuss a number of methods and we give references to them for the benefit of the reader. We first introduce some notation in order to promote the understandability of what is to follow:

$L^{h,k} =$ fully discrete approximate to L
$I^h =$ discrete approximation to the integral term $I(u)$

We do not further specify the exact kinds of approximations for the integral, but they could be:

- Trapezoidal integration rule
- The tanh integration rule (Duffy 2004)

In the time dimension, we apply exclusively one-step methods and then we can choose between fully explicit, fully implicit and Crank Nicolson variants (as usual). In the following discussion we suppress the dependence of the discrete solution on the index for the S direction. This makes the schemes a bit more readable.

A. Implicit and Explicit Methods

A simple approach is to apply the so-called theta-method for both the PDE and integral terms:

$$\begin{aligned} &\tfrac{U^{n+1}-U^n}{k} = \theta_1 L^{h,k}(U^n) + (1-\theta_1) L^{h,k}(U^{n+1}) \\ &+\theta_2 I^h(U^n) + (1-\theta_2) I^h(U^{n+1}) \end{aligned} \tag{9}$$

This system of equations leads to a dense matrix system in general. This can be remedied by approximating the integral term only at the known time level n. For example, using Crank Nicolson for the PDE term and integral evaluation at level n leads to the equation:

$$\tfrac{U^{n+1}-U^n}{k} = \tfrac{1}{2}\left\{L^{h,k}(U^n) + L^{h,k}(U^{n+1})\right\} + I^h(u^n) \tag{10}$$

This is an interesting scheme because it is a tridiagonal matrix system and it has been proved that this scheme is conditionally stable (see Cont 2003). The authors prove stability and accuracy of the scheme (10) using the so-called viscosity method. This is needed because the famous and sometimes misused Lax Equivalence Principle is no longer valid due to the fact that solutions of the PIDE may be non-smooth and higher-order derivatives may not exist. The article Cont 2003 is well worth reading because it discusses the current problem in a thorough manner using modern mathematical techniques.

B. Implicit-Explicit Runge-Kutta (IMEX) Methods

It is obvious that the coupling between the differential and integral terms in equation (5) complicates the discovery of suitable numerical schemes and subsequent analysis. If would be nice if we could split the problem in some way so that we could solve several simpler sub-problems. To this end, we introduce a method that splits the PIDE in such a way that one part is implicit in time and the other part is explicit in time. The method is called the Implicit-Explicit (IMEX) and the rationale behind it is to split a scheme into its stiff and non-stiff parts (see Hundsdorfer 2003 for a good introduction to IMEX methods). In the current case we define the following 'components' of the PIDE (5):

$$
\begin{aligned}
&\tfrac{\partial u}{\partial t} = \sigma \tfrac{\partial^2 u}{\partial x^2} + \mu \tfrac{\partial u}{\partial x} + bu + \lambda \left\{ \int_{-\infty}^{\infty} u(x+y,t)\Gamma_\sigma(y)dy - u(x,t) \right\} \\
&\tfrac{\partial u}{\partial t} = H(u) + G(u) \\
&H(u) = \mu \tfrac{\partial u}{\partial x} + \lambda \left\{ \int_{-\infty}^{\infty} u(x+y,t)\Gamma_\sigma(y)dy - u(x,t) \right\} \\
&G(u) = \sigma \tfrac{\partial^2 u}{\partial x^2} + bu
\end{aligned}
\tag{11}
$$

In this case we use implicit time-stepping for the diffusion term G(u) and explicit time-stepping for the convection (advection) term H(u). Then one particular IMEX scheme for this problem becomes:

$$
\frac{U^{n+1} - U^n}{k} = H(U^n) + \theta\, G(U^n) + (1-\theta) G(U^{n+1}), \quad \theta \geq \frac{1}{2} \tag{12}
$$

Here we see that the Euler method is combined with an A-stable theta method (Hunsdorfer 2003). A generalisation of this method is given in Briani 2004 where the authors propose a multistep variant of equation (12).

C. Using Operator Splitting (Method of Fractional Steps)

Operator splitting methods are a powerful technique for splitting a problem into simpler ones. In general, an n-dimensional problem can be split into a series of one-dimensional problems using this method. In the current context, operator splitting has been applied to integro-differential equations for the neutron transport problem (see Yanenko 1971, page 99). We already know that the PIDE (5) can be split into a differential and integral form:

$$
\frac{\partial u}{\partial t} = Lu + Iu \tag{13}
$$

Based in this remark we split the problem into two sub-problems that we write as (in an intuitive/semi-formal form):

$$
\begin{aligned}
&\tfrac{\partial u}{\partial t} = Iu \\
&\tfrac{\partial u}{\partial t} = Lu
\end{aligned}
\tag{14}
$$

Based on Yanenko 1971 we propose the following splitting scheme:

$$\begin{array}{l}(a)\frac{U^{n+1/2}-U^n}{k} = I^h(\alpha\, U^{n+1/2} + \beta U^n)\\ (b)\frac{U^{n+1}-U^{n+1/2}}{k} = L^h(\alpha\, U^{n+1} + \beta U^{n+1/2})\\ \text{where}\\ \alpha \geq 0, \beta \geq 0, \alpha+\beta = 1\end{array} \tag{15}$$

We can choose different values of α and β to give use implicit or explicit schemes. For example, we could take an explicit scheme for (15)(a) and the exponentially fitted implicit-in-time scheme for (15)(b):

$$\begin{array}{l}(a)\ \frac{U^{n+1/2}-U^n}{k} = I^h(U^n)\\ (b)\ \frac{U^{n+1}-U^{n+1/2}}{k} = L_E^{h,k}(U^{n+1})\\ \text{where}\\ L_E^{h,k} \equiv \text{(Duffy) Exporential fitting operator}\end{array} \tag{16}$$

See Duffy 2004 for a detailed analysis of exponentially fitted finite difference schemes.

D. Splitting in conjunction with Predictor-Corrector Methods

We now discuss the following problem: can we devise a scheme that has the computational ease of the explicit Euler scheme while at the same time achieving high-order accuracy? We answer this question by appealing to the predictor-corrector methods that is used in the approximation of initial value problems (see Conte 1980, page 379) and that we have applied with success to financial engineering applications, in particular to the numerical solution of deterministic and stochastic differential equations (Duffy 2004). Let us recall how predictor-corrector works for the initial value problem:

$$\begin{array}{l}\frac{du}{dt} = f(t,u),\ 0 < t < T\\ u(0) = A\\ u(t) = {}^t(u_1(t), \ldots, u_n(t))\end{array} \tag{17}$$

If we apply the standard trapezoidal rule to (17) we get the scheme:

$$\frac{u_{n+1} - u_n}{k} = \frac{1}{2}\left\{f(t_n, u_n) + f(t_{n+1}, u_{n+1})\right\} \tag{18}$$

The only problem with this scheme is that it is nonlinear since the function f(t, u) is in general a nonlinear function in u. Hence the system (18) cannot be solved without resorting to some method such as Newton-Raphson, for example. In order to resolve this problem we define predictor and corrector solutions as follows:

$$\begin{array}{l} u_{n+1}^{(0)} = u_n + kf(t_n, u_n) \\ u_{n+1}^{(1)} = u_n + \frac{k}{2}[f(t_n, u_n) + f(t_{n+1}, u_{n+1}^{(0)})] \end{array} \tag{19}$$

This is the essence of predictor-corrector method. In general, we define the iterative scheme:

$$u_{n+1}^{(j)} = u_n + \frac{k}{2}[f(t_n, u_n) + f(t_{n+1}, u_{n+1}^{(j-1)})] \;\; j = 1, 2, \ldots \tag{20}$$

and the stopping criterion for a given tolerance TOL is given by

$$\frac{\|u_{n+1}^{(j)} - u_{n+1}^{(j-1)}\|}{\|u_{n+1}^{(j)}\|} \leq TOL \tag{21}$$

in some suitable norm (for example, the max norm).

We now apply the predictor-corrector method to generalise the scheme (16)(a). We define:

$$\begin{array}{l} U_{n+1}^{(0)} = U_n + kI^h(U_n) \\ \\ U_{n+1}^{(1)} = U_n + \frac{k}{2}[I^h(U_n) + I^h(U_{n+1}^{(0)})] \\ \text{more generally} \\ \\ U_{n+1}^{(j)} = U_n + \frac{k}{2}[I^h(U_n) + I^h(U_{n+1}^{(j-1)})] \;\; j = 1, 2, \ldots \end{array} \tag{22}$$

with the same stopping criteria as in inequality (22).

A more detailed account of numerical methods for PIDE can be found in Duffy 2005. An interesting discussion of ADI in conjunction with the discrete Fourier transform as a means of solving PIDEs is given in Andersen 2000.

Conclusions

We have given an overview of a number of numerical methods that approximate the solution of the partial integro-differential equations (PIDE) that model jump diffusion models in option pricing. Since there is no closed solution in general we must resort to approximate methods. Some conclusions on the advantages and disadvantages of the finite difference methods are:

- Explicit and Implicit: easy to implement as it builds on well-known schemes. Conditionally stable. First-order accurate.
- IMEX: robust, modern schemes. Ability to handle stiff problems. Second and higher order accuracy.
- Operator Splitting: reliable and robust. Watch out for the errors induced by splitting. You may not always get second order accuracy.
- Predictor-Corrector: a good performer as has been proved in many applications. Applicable to no-linear kernels.

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