Exercises 1.1

Preliminary Reading

Lipschutz, chapters 1, 2; Apostol, parts 2,3

Sets and Relations

1. Find the *power set* (set of all subsets) of the sets $1, 2, 3$ and $1, 2, 3, 4, 5$. How many subsets are there?

   Do the answers give a clue as to the number of subsets of a set with n elements? Prove your conjecture using *mathematical induction* (Lipschutz page 12).

2. Consider the power set for $1, 2, 3, 4$. Enumerate the subsets of the power set containing k elements, $k = 0, 1, 2, 3, 4$. How many elements are there in each subset? Can you generalize the result: how many subsets of size k are there in a set with n elements $(n > k)$?

3. Prove the following results using mathematical induction:

   a) In a population of n people, there are $n(n-1)/2$ possible (undirected) communication paths.

   b) $1 + 2 + 3 + .. + n = n(n+1)/2$

   c) $\sum_{k=0}^{n} \binom{n}{k} = 2^n, \quad n > k$ (see Apostol page 44)

4. Find the *Cartesian product* of the sets A and B, where $A = 1, 2, 3, B = 1, 2, 3, 4$. How many elements does it have? How many elements does the Cartesian product have in the case when A has n elements and B has m elements?

5. Consider the mapping $f(x) = x^2$ from a set A to itself. Determine if the mapping is *injective, surjective, bijective* in the following cases:

   - A = set of all integers.
   - A = set of positive integers.
   - A = set of real numbers.
   - A = set of complex numbers (have *real* and imaginary parts).

6. Consider tossing a fair coin three times. The outcome per throw is either Head(H) or Tail (T). Answers the following questions:

   a) What is the *sample space* of this experiment, that is the set of all possible outcomes?

   b) What is the set corresponding to finding outcomes containing two heads?

   c) Consider tossing a fair coin four times. Describe the sample space. (this exercise is related to probability theory that we discuss in a later section).

Numerical Sequences and Series

7. Write (or compute) the first 10 terms of the following sequences:

$$y_n = \frac{1}{(n+2)^3}, \quad y_n = \frac{n+1}{n+2}$$

$$y_n = \sin\frac{n\pi}{2}, \quad y_n = \frac{n+(-1)^n}{n+1}$$

Are these sequences converging, diverging, monotonic, oscillating, bounded?

8. Consider the sequence defined by the one-step method:

$$y_{n+1} = Ay_n + B, \quad n = 0, 1, 2, ..$$

Prove that

$$y_n = \begin{cases} A^n y_0 + B\frac{1-A^n}{1-A}, & A \neq 1 \\ \\ y_0 + B, & A = 1 \text{ for } n = 0, 1, 2, .. \end{cases}$$

9. Prove that the sequence $\left\{\frac{1}{n}\right\}, n \geq 1$ is a Cauchy sequence. Prove that a convergent sequence is also a Cauchy sequence. Give an example of a non-convergent Cauchy sequence.

10. Apply the *root* test and *ratio* test to determine the convergence (or otherwise) of the following series:

$$\sum \frac{1}{n^2}, \sum \frac{1}{n}, \sum \frac{n}{e^n}$$

11. Prove the following alternating series converge:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \log 2$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \pi/4$$

Continuity

12. Prove that the following function is continuous at $x = 0$:

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Draw a graph of the function (by any means).

13. Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{R}$ the set of real numbers.

    Define the function:

$$f(x) = \begin{cases} 0, & x \in \mathbb{R} \backslash \mathbb{Q} \\ \\ x, & x \in \mathbb{Q} \end{cases}$$

    Prove that f is continuous at only one point, namely $x = 0$.

14. Define the *Heaviside function*

$$H(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ 1, & x > 0 \end{cases}$$

    What kind of discontinuity do we have at $x = 0$?

Limits and Functions

15. Use l'Hopital's rule to find the following limits (you may have to differentiate several times in some cases):

$$\lim_{x \to 0} \frac{2 \sin x - \sin 2x}{x - \sin x}, \quad \lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

$$\lim_{x \to 0+} x \log x, \quad \lim \frac{b^x - 1}{x}$$

16. Use l'Hopital's rule to find the following limit (hint: use a log transformation to simplify the function):

$$\lim_{x \to \infty} \left(1 + \frac{4}{x}\right)^x$$

17. Use the Weierstrass M-test to prove the uniform convergence of the series:

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{3^n} \cos\left(\frac{x}{2^n}\right)$$